RUSSIAN FEDERAL COMMITTEE
FOR HIGHER EDUCATION

BASHKIR STATE UNIVERSITY

SHARIPOV R. A.

COURSE OF DIFFERENTIAL GEOMETRY

The Textbook

Ufa 1996

MSC 97U20
UDC 514.7

Sharipov R. A. **Course of Differential Geometry**: the textbook / Publ. of Bashkir State University — Ufa, 1996. — pp. 132. — ISBN 5-7477-0129-0.

This book is a textbook for the basic course of differential geometry. It is recommended as an introductory material for this subject.

In preparing Russian edition of this book I used the computer typesetting on the base of the $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX package and I used Cyrillic fonts of the Lh-family distributed by the CyrTUG association of Cyrillic TEX users. English edition of this book is also typeset by means of the $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX package.

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ISBN 5-7477-0129-0
English translation

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PREFACE.

This book was planned as the third book in the series of three textbooks for three basic geometric disciplines of the university education. These are

– «Course of analytical geometry[1]»;
– «Course of linear algebra and multidimensional geometry»;
– «Course of differential geometry».

This book is devoted to the first acquaintance with the differential geometry. Therefore it begins with the theory of curves in three-dimensional Euclidean space $\mathbb{E}$. Then the vectorial analysis in $\mathbb{E}$ is stated both in Cartesian and curvilinear coordinates, afterward the theory of surfaces in the space $\mathbb{E}$ is considered.

The newly fashionable approach starting with the concept of a differentiable manifold, to my opinion, is not suitable for the introduction to the subject. In this way too many efforts are spent for to assimilate this rather abstract notion and the rather special methods associated with it, while the the essential content of the subject is postponed for a later time. I think it is more important to make faster acquaintance with other elements of modern geometry such as the vectorial and tensorial analysis, covariant differentiation, and the theory of Riemannian curvature. The restriction of the dimension to the cases $n = 2$ and $n = 3$ is not an essential obstacle for this purpose. The further passage from surfaces to higher-dimensional manifolds becomes more natural and simple.

I am grateful to D. N. Karbushev, R. R. Bakhitov, S. Yu. Ubiyko, D. I. Borisov (http://borisovdi.narod.ru), and Yu. N. Polyakov for reading and correcting the manuscript of the Russian edition of this book.

November, 1996;
December, 2004. R. A. Sharipov.

[1] Russian versions of the second and the third books were written in 1096, but the first book is not yet written. I understand it as my duty to complete the series, but I had not enough time all these years since 1996.

CHAPTER I

CURVES IN THREE-DIMENSIONAL SPACE.

§ 1. Curves. Methods of defining a curve. Regular and singular points of a curve.

Let $\mathbb{E}$ be a three-dimensional Euclidean point space. The strict mathematical definition of such a space can be found in [1]. However, knowing this definition is not so urgent. The matter is that $\mathbb{E}$ can be understood as the regular three-dimensional space (that in which we live). The properties of the space $\mathbb{E}$ are studied in elementary mathematics and in analytical geometry on the base intuitively clear visual forms. The concept of a *line* or a *curve* is also related to some visual form. A *curve* in the space $\mathbb{E}$ is a spatially extended one-dimensional geometric form. The one-dimensionality of a curve reveals when we use the *vectorial-parametric* method of defining it:

$$\mathbf{r} = \mathbf{r}(t) = \begin{Vmatrix} x^1(t) \\ x^2(t) \\ x^3(t) \end{Vmatrix}. \tag{1.1}$$

We have one degree of freedom when choosing a point on the curve (1.1), our choice is determined by the value of the numeric parameter t taken from some interval, e. g. from the unit interval $[0,\,1]$ on the real axis $\mathbb{R}$. Points of the curve (1.1) are given by their radius-vectors[1] $\mathbf{r} = \mathbf{r}(t)$ whose components $x^1(t)$, $x^2(t)$, $x^3(t)$ are functions of the parameter t.

The continuity of the curve (1.1) means that the functions $x^1(t)$, $x^2(t)$, $x^3(t)$ should be continuous. However, this condition is too weak. Among continuous curves there are some instances which do not agree with our intuitive understanding of a curve. In the course of mathematical analysis the Peano curve is often considered as an example (see [2]). This is a continuous parametric curve on a plane such that it is enclosed within a unit square, has no self intersections, and passes through each point of this square. In order to avoid such unusual curves the functions $x^i(t)$ in (1.1) are assumed to be continuously differentiable (C^1 class) functions or, at least, piecewise continuously differentiable functions.

Now let's consider another method of defining a curve. An arbitrary point of the space $\mathbb{E}$ is given by three arbitrary parameters x^1, x^2, x^3 — its coordinates. We can restrict the degree of arbitrariness by considering a set of points whose coordinates x^1, x^2, x^3 satisfy an equation of the form

$$F(x^1, x^2, x^3) = 0, \tag{1.2}$$

[1] Here we assume that some Cartesian coordinate system in $\mathbb{E}$ is taken.

where F is some continuously differentiable function of three variables. In a typical situation formula (1.2) still admits two-parametric arbitrariness: choosing arbitrarily two coordinates of a point, we can determine its third coordinate by solving the equation (1.2). Therefore, (1.2) is an equation of a surface. In the intersection of two surfaces usually a curve arises. Hence, a system of two equations of the form (1.2) defines a curve in $\mathbb{E}$:

$$\begin{cases} F(x^1, x^2, x^3) = 0, \\ G(x^1, x^2, x^3) = 0. \end{cases} \tag{1.3}$$

If a curve lies on a plane, we say that it is a *plane* curve. For a plane curve one of the equations (1.3) can be replaced by the equation of a plane: $A\,x^1 + B\,x^2 + C\,x^3 + D = 0$.

Suppose that a curve is given by the equations (1.3). Let's choose one of the variables x^1, x^2, or x^3 for a parameter, e. g. we can take $x^1 = t$ to make certain. Then, writing the system of the equations (1.3) as

$$\begin{cases} F(t, x^2, x^3) = 0, \\ G(t, x^2, x^3) = 0, \end{cases}$$

and solving them with respect to x^2 and x^3, we get two functions $x^2(t)$ and $x^3(t)$. Hence, the same curve can be given in vectorial-parametric form:

$$\mathbf{r} = \mathbf{r}(t) = \begin{Vmatrix} t \\ x^2(t) \\ x^3(t) \end{Vmatrix}.$$

Conversely, assume that a curve is initially given in vectorial-parametric form by means of vector-functions (1.1). Then, using the functions $x^1(t)$, $x^2(t)$, $x^3(t)$, we construct the following two systems of equations:

$$\begin{cases} x^1 - x^1(t) = 0, \\ x^2 - x^2(t) = 0, \end{cases} \qquad\qquad \begin{cases} x^1 - x^1(t) = 0, \\ x^3 - x^3(t) = 0. \end{cases} \tag{1.4}$$

Excluding the parameter t from the first system of equations (1.4), we obtain some functional relation for two variable x^1 and x^2. We can write it as $F(x^1, x^2) = 0$. Similarly, the second system reduces to the equation $G(x^1, x^3) = 0$. Both these equations constitute a system, which is a special instance of (1.3):

$$\begin{cases} F(x^1, x^2) = 0, \\ G(x^1, x^3) = 0. \end{cases}$$

This means that the vectorial-parametric representation of a curve can be transformed to the form of a system of equations (1.3).

None of the above two methods of defining a curve in $\mathbb{E}$ is absolutely preferable. In some cases the first method is better, in other cases the second one is used. However, for constructing the theory of curves the vectorial-parametric method is more suitable. Suppose that we have a parametric curve γ of the *smoothness class* C^1. This is a curve with the coordinate functions $x^1(t)$, $x^2(t)$, $x^3(t)$ being continuously differentiable. Let's choose two different values of the parameter: t and $\tilde{t} = t + \triangle t$, where $\triangle t$ is an increment of the parameter. Let A and B be two points on the curve corresponding to that two values of the parameter t. We draw the straight line passing through these points A and B; this is a secant for the curve γ. Directing vectors of this secant are collinear to the vector $\overrightarrow{AB}$. We choose one of them:

$$\mathbf{a} = \frac{\overrightarrow{AB}}{\triangle t} = \frac{\mathbf{r}(t + \triangle t) - \mathbf{r}(t)}{\triangle t}. \tag{1.5}$$

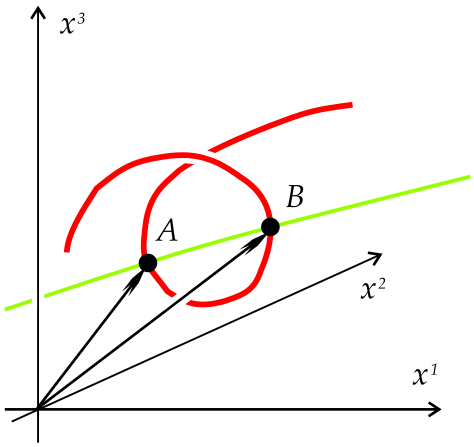


Fig. 1.1

Tending $\triangle t$ to zero, we find that the point B moves toward the point A. Then the secant tends to its limit position and becomes the tangent line of the curve at the point A. Therefore limit value of the vector (1.5) is a tangent vector of the curve γ at the point A:

$$\boldsymbol{\tau}(t) = \lim_{\triangle t \to \infty} \mathbf{a} = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t). \tag{1.6}$$

The components of the tangent vector (1.6) are evaluated by differentiating the components of the radius-vector $\mathbf{r}(t)$ with respect to the variable t.

The tangent vector $\dot{\mathbf{r}}(t)$ determines the direction of the instantaneous displacement of the point $\mathbf{r}(t)$ for the given value of the parameter t. Those points, at which the derivative $\dot{\mathbf{r}}(t)$ vanishes, are special ones. They are «stopping points». Upon stopping, the point can begin moving in quite different direction. For example, let's consider the following two plane curves:

$$\mathbf{r}(t) = \begin{Vmatrix} t^2 \\ t^3 \end{Vmatrix}, \qquad\qquad \mathbf{r}(t) = \begin{Vmatrix} t^4 \\ t^3 \end{Vmatrix}. \tag{1.7}$$

At $t = 0$ both curves (1.7) pass through the origin and the tangent vectors of both curves at the origin are equal to zero. However, the behavior of these curves near the origin is quite different: the first curve has a beak-like fracture at the origin,

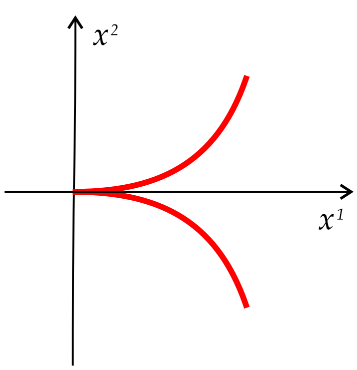


Fig. 1.2

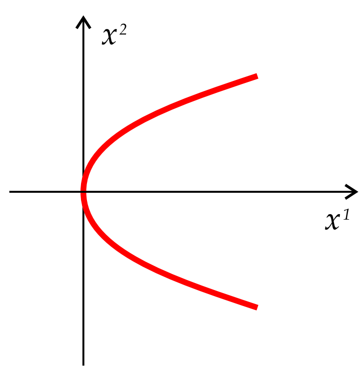


Fig. 1.3

while the second one is smooth. Therefore, vanishing of the derivative

$$\boldsymbol{\tau}(t) = \dot{\mathbf{r}}(t) = 0 \tag{1.8}$$

is only the necessary, but not sufficient condition for a parametric curve to have a singularity at the point $\mathbf{r}(t)$. The opposite condition

$$\boldsymbol{\tau}(t) = \dot{\mathbf{r}}(t) \neq 0 \tag{1.9}$$

guaranties that the point $\mathbf{r}(t)$ is free of singularities. Therefore, those points of a parametric curve, where the condition (1.9) is fulfilled, are called *regular points*.

Let's study the problem of separating regular and singular points on a curve given by a system of equations (1.3). Let $A = (a^1, a^2, a^3)$ be a point of such a curve. The functions $F(x^1, x^2, x^3)$ and $G(x^1, x^2, x^3)$ in (1.3) are assumed to be continuously differentiable. The matrix

$$J = \begin{Vmatrix} \dfrac{\partial F}{\partial x^1} & \dfrac{\partial F}{\partial x^2} & \dfrac{\partial F}{\partial x^3} \\[2ex] \dfrac{\partial G}{\partial x^1} & \dfrac{\partial G}{\partial x^2} & \dfrac{\partial G}{\partial x^3} \end{Vmatrix} \tag{1.10}$$

composed of partial derivatives of F and G at the point A is called the *Jacobi* matrix or the *Jacobian* of the system of equations (1.3). If the minor

$$M_1 = \det \begin{vmatrix} \dfrac{\partial F}{\partial x^2} & \dfrac{\partial F}{\partial x^3} \\[2ex] \dfrac{\partial G}{\partial x^2} & \dfrac{\partial G}{\partial x^3} \end{vmatrix} \neq 0$$

in Jacobi matrix is nonzero, the equations (1.3) can be resolved with respect to x^2 and x^3 in some neighborhood of the point A. Then we have three functions $x^1 = t$, $x^2 = x^2(t)$, $x^3 = x^3(t)$ which determine the parametric representation of our curve. This fact follows from the theorem *on implicit functions* (see [2]). Note that the tangent vector of the curve in this parametrization

$$\boldsymbol{\tau} = \begin{Vmatrix} 1 \\ \dot{x}^2 \\ \dot{x}^3 \end{Vmatrix} \neq 0$$

is nonzero because of its first component. This means that the condition $M_1 \neq 0$ is sufficient for the point A to be a regular point of a curve given by the system of equations (1.3). Remember that the Jacobi matrix (1.10) has two other minors:

$$M_2 = \det \begin{vmatrix} \dfrac{\partial F}{\partial x^3} & \dfrac{\partial F}{\partial x^1} \\[2ex] \dfrac{\partial G}{\partial x^3} & \dfrac{\partial G}{\partial x^1} \end{vmatrix}, \qquad M_3 = \det \begin{vmatrix} \dfrac{\partial F}{\partial x^1} & \dfrac{\partial F}{\partial x^2} \\[2ex] \dfrac{\partial G}{\partial x^1} & \dfrac{\partial G}{\partial x^2} \end{vmatrix}.$$

For both of them the similar propositions are fulfilled. Therefore, we can formulate the following theorem.

Theorem 1.1. *A curve given by a system of equations* (1.3) *is regular at all points, where the rank of its Jacobi matrix* (1.10) *is equal to* 2.

A plane curve lying on the plane $x^3 = 0$ can be defined by one equation $F(x^1, x^2) = 0$. The second equation here reduces to $x^3 = 0$. Therefore, $G(x^1, x^2, x^3) = x^3$. The Jacoby matrix for the system (1.3) in this case is

$$J = \left\| \begin{matrix} \dfrac{\partial F}{\partial x^1} & \dfrac{\partial F}{\partial x^2} & 0 \\ 0 & 0 & 1 \end{matrix} \right\|. \tag{1.11}$$

If $\operatorname{rank} J = 2$, this means that at least one of two partial derivatives in the matrix (1.11) is nonzero. These derivatives form the *gradient* vector for the function F:

$$\operatorname{grad} F = \left(\frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2}\right).$$

Theorem 1.2. *A plane curve given by an equation* $F(x^1, x^2) = 0$ *is regular at all points where* $\operatorname{grad} F \neq 0$.

This theorem 1.2 is a simple corollary from the theorem 1.1 and the relationship (1.11). Note that the theorems 1.1 and 1.2 yield only sufficient conditions for regularity of curve points. Therefore, some points where these theorems are not applicable can also be regular points of a curve.

§ 2. The length integral and the natural parametrization of a curve.

Let $\mathbf{r} = \mathbf{r}(t)$ be a parametric curve of smoothness class C^1, where the parameter t runs over the interval $[a, b]$. Let's consider a monotonic increasing continuously differentiable function $\varphi(\tilde{t})$ on a segment $[\tilde{a}, \tilde{b}]$ such that $\varphi(\tilde{a}) = a$ and $\varphi(\tilde{b}) = b$. Then it takes each value from the segment $[a, b]$ exactly once. Substituting $t = \varphi(\tilde{t})$ into $\mathbf{r}(t)$, we define the new vector-function $\tilde{\mathbf{r}}(\tilde{t}) = \mathbf{r}(\varphi(\tilde{t}))$, it describes the same curve as the original vector-function $\mathbf{r}(t)$. This procedure is called the *reparametrization* of a curve. We can calculate the tangent vector in the new parametrization by means of the chain rule:

$$\tilde{\boldsymbol{\tau}}(\tilde{t}) = \varphi'(\tilde{t}) \cdot \boldsymbol{\tau}(\varphi(\tilde{t})). \tag{2.1}$$

Here $\varphi'(\tilde{t})$ is the derivative of the function $\varphi(\tilde{t})$. The formula (2.1) is known as the *transformation rule* for the tangent vector of a curve under a change of parametrization.

A monotonic decreasing function $\varphi(\tilde{t})$ can also be used for the reparametrization of curves. In this case $\varphi(\tilde{a}) = b$ and $\varphi(\tilde{b}) = a$, i.e. the beginning point and the ending point of a curve are exchanged. Such reparametrizations are called *changing the orientation of a curve.*

From the formula (2.1), we see that the tangent vector $\tilde{\boldsymbol{\tau}}(\tilde{t})$ can vanish at some points of the curve due to the derivative $\varphi'(\tilde{t})$ even when $\boldsymbol{\tau}(\varphi(\tilde{t}))$ is nonzero.

Certainly, such points are not actually the singular points of a curve. In order to exclude such formal singularities, only those reparametrizations of a curve are admitted for which the function $\varphi(\tilde{t})$ is a strictly monotonic function, i.e. $\varphi'(\tilde{t}) > 0$ or $\varphi'(\tilde{t}) < 0$.

The formula (2.1) means that the tangent vector of a curve at its regular point depends not only on the geometry of the curve, but also on its parametrization. However, the effect of parametrization is not so big, it can yield a numeric factor to the vector $\boldsymbol{\tau}$ only. Therefore, the natural question arises: is there some preferable parametrization on a curve? The answer to this question is given by the length integral.

Let's consider a segment of a parametric curve of the smoothness class C^1 with the parameter t running over the segment $[a, b]$ of real numbers. Let

$$a = t_0 < t_1 < \ldots < t_n = b \tag{2.2}$$

be a series of points breaking this segment into n parts. The points $\mathbf{r}(t_0), \ldots, \mathbf{r}(t_n)$ on the curve define a polygonal line with n segments. Denote $\triangle t_k = t_k - t_{k-1}$ and let ε be the maximum of $\triangle t_k$:

$$\varepsilon = \max_{k=1,\ldots,n} \triangle t_k.$$

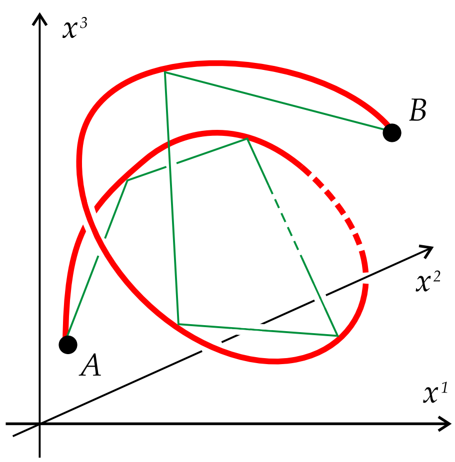


Fig. 2.1

The quantity ε is the fineness of the partition (2.2). The length of k-th segment of the polygonal line AB is calculated by the formula $L_k = |\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})|$. Using the continuous differentiability of the vector-function $\mathbf{r}(t)$, from the Taylor expansion of $\mathbf{r}(t)$ at the point t_{k-1} we get $L_k = |\boldsymbol{\tau}(t_{k-1})| \cdot \triangle t_k + o(\varepsilon)$. Therefore, as the fineness ε of the partition (2.2) tends to zero, the length of the polygonal line AB has the limit equal to the integral of the modulus of tangent vector $\boldsymbol{\tau}(t)$ along the curve:

$$L = \lim_{\varepsilon\to 0} \sum_{k=1}^{n} L_k = \int_a^b |\boldsymbol{\tau}(t)|\,dt. \tag{2.3}$$

It is natural to take the quantity L in (2.3) for the length of the curve AB. Note that if we reparametrize a curve according to the formula (2.1), this leads to a change of variable in the integral. Nevertheless, the value of the integral L remains unchanged. Hence, the length of a curve is its geometric invariant which does not depend on the way how it is parameterized.

The length integral (2.3) defines the preferable way for parameterizing a curve in the Euclidean space $\mathbb{E}$. Let's denote by $s(t)$ an antiderivative of the function

$\psi(t) = |\boldsymbol{\tau}(t)|$ being under integration in the formula (2.3):

$$s(t) = \int\limits_{t_0}^{t} |\boldsymbol{\tau}(t)|\,dt. \tag{2.4}$$

Definition 2.1. The quantity s determined by the integral (2.4) is called the *natural parameter* of a curve in the Euclidean space $\mathbb{E}$.

Note that once the reference point $\mathbf{r}(t_0)$ and some direction (orientation) on a curve have been chosen, the value of natural parameter depends on the point of the curve only. Then the change of s for $-s$ means the change of orientation of the curve for the opposite one.

Let's differentiate the integral (2.4) with respect to its upper limit t. As a result we obtain the following relationship:

$$\frac{ds}{dt} = |\boldsymbol{\tau}(t)|. \tag{2.5}$$

Now, using the formula (2.5), we can calculate the tangent vector of a curve in its natural parametrization, i. e. when s is used instead of t as a parameter:

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\mathbf{r}}{dt} \bigg/ \frac{ds}{dt} = \frac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|}. \tag{2.6}$$

From the formula (2.6), we see that in the tangent vector of a curve in natural parametrization is a unit vector at all regular points. In singular points this vector is not defined at all.

§ 3. Frenet frame. The dynamics of Frenet frame. Curvature and torsion of a spacial curve.

Let's consider a smooth parametric curve $\mathbf{r}(s)$ in natural parametrization. The components of the radius-vector $\mathbf{r}(s)$ for such a curve are smooth functions of s (smoothness class C^∞). They are differentiable unlimitedly many times with respect to s. The unit vector $\boldsymbol{\tau}(s)$ is obtained as the derivative of $\mathbf{r}(s)$:

$$\boldsymbol{\tau}(s) = \frac{d\mathbf{r}}{ds}. \tag{3.1}$$

Let's differentiate the vector $\boldsymbol{\tau}(s)$ with respect to s and then apply the following lemma to its derivative $\boldsymbol{\tau}'(s)$.

Lemma 3.1. *The derivative of a vector of a constant length is a vector perpendicular to the original one.*

Proof. In order to prove the lemma we choose some standard rectangular Cartesian coordinate system in $\mathbb{E}$. Then

$$|\boldsymbol{\tau}(s)|^2 = (\boldsymbol{\tau}(s)\,|\,\boldsymbol{\tau}(s)) = (\tau^1)^2 + (\tau^2)^2 + (\tau^3)^2 = \mathrm{const}\,.$$

Let's differentiate this expression with respect to s. As a result we get the following relationship:

$$\begin{aligned}\frac{d}{ds}\left(|\boldsymbol{\tau}(s)|^2\right) = \frac{d}{ds}\left((\tau^1)^2 + (\tau^2)^2 + (\tau^3)^2\right) &=\\ = 2\,\tau^1\,(\tau^1)' + 2\,\tau^2\,(\tau^2)' + 2\,\tau^3\,(\tau^3)' &= 0.\end{aligned}$$

One can easily see that this relationship is equivalent to $(\boldsymbol{\tau}(s)\,|\,\boldsymbol{\tau}'(s)) = 0$. Hence, $\boldsymbol{\tau}(s) \perp \boldsymbol{\tau}'(s)$. The lemma is proved. $\square$

Due to the above lemma the vector $\boldsymbol{\tau}'(s)$ is perpendicular to the unit vector $\boldsymbol{\tau}(s)$. If the length of $\boldsymbol{\tau}'(s)$ is nonzero, one can represent it as

$$\boldsymbol{\tau}'(s) = k(s)\cdot\mathbf{n}(s), \tag{3.2}$$

where $k(s) = |\boldsymbol{\tau}'(s)|$ and $|\mathbf{n}(s)| = 1$. The scalar quantity $k(s) = |\boldsymbol{\tau}'(s)|$ in formula (3.2) is called the *curvature* of a curve, while the unit vector $\mathbf{n}(s)$ is called its *primary normal vector* or simply the *normal vector* of a curve at the point $\mathbf{r}(s)$. The unit vectors $\boldsymbol{\tau}(s)$ and $\mathbf{n}(s)$ are orthogonal to each other. We can complement them by the third unit vector $\mathbf{b}(s)$ so that $\boldsymbol{\tau}$, $\mathbf{n}$, $\mathbf{b}$ become a *right triple*[1]:

$$\mathbf{b}(s) = [\boldsymbol{\tau}(s),\,\mathbf{n}(s)]. \tag{3.3}$$

The vector $\mathbf{b}(s)$ defined by the formula (3.3) is called the *secondary normal* vector or the *binormal* vector of a curve. Vectors $\boldsymbol{\tau}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$ compose an orthonormal right basis attached to the point $\mathbf{r}(s)$.

Bases, which are attached to some points, are usually called *frames*. One should distinguish frames from coordinate systems. Cartesian coordinate systems are also defined by choosing some point (an origin) and some basis. However, coordinate systems are used for describing the points of the space through their coordinates. The purpose of frames is different. They are used for to expand the vectors which, by their nature, are attached to the same points as the vectors of the frame.

The isolated frames are rarely considered, frames usually arise within families of frames: typically at each point of some set (a curve, a surface, or even the whole space) there arises some frame attached to this point. The frame $\boldsymbol{\tau}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$ is an example of such frame. It is called the *Frenet frame* of a curve. This is the moving frame: in typical situation the vectors of this frame change when we move the attachment point along the curve.

Let's consider the derivative $\mathbf{n}'(s)$. This vector attached to the point $\mathbf{r}(s)$ can be expanded in the Frenet frame at that point. Due to the lemma 3.1 the vector $\mathbf{n}'(s)$ is orthogonal to the vector $\mathbf{n}(s)$. Therefore its expansion has the form

$$\mathbf{n}'(s) = \alpha\cdot\boldsymbol{\tau}(s) + \varkappa\cdot\mathbf{b}(s). \tag{3.4}$$

The quantity α in formula (3.4) can be expressed through the curvature of the

[1] A non-coplanar ordered triple of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ is called a *right triple* if, upon moving these vectors to a common origin, when looking from the end of the third vector $\mathbf{a}_3$, we see the shortest rotation from $\mathbf{a}_1$ to $\mathbf{a}_2$ as a counterclockwise rotation.

curve. Indeed, as a result of the following calculations we derive

$$\begin{aligned}\alpha(s) &= (\boldsymbol{\tau}(s)\,|\,\mathbf{n}'(s)) = (\boldsymbol{\tau}(s)\,|\,\mathbf{n}(s))' - \\ &- (\boldsymbol{\tau}'(s)\,|\,\mathbf{n}(s)) = -(k(s)\cdot\mathbf{n}(s)\,|\,\mathbf{n}(s)) = -k(s).\end{aligned}\tag{3.5}$$

The quantity $\varkappa = \varkappa(s)$ cannot be expressed through the curvature. This is an additional parameter characterizing a curve in the space $\mathbb{E}$. It is called the *torsion* of the curve at the point $\mathbf{r} = \mathbf{r}(s)$. The above expansion (3.4) of the vector $\mathbf{n}'(s)$ now is written in the following form:

$$\mathbf{n}'(s) = -k(s)\cdot\boldsymbol{\tau}(s) + \varkappa(s)\cdot\mathbf{b}(s).\tag{3.6}$$

Let's consider the derivative of the binormal vector $\mathbf{b}'(s)$. It is perpendicular to $\mathbf{b}(s)$. This derivative can also be expanded in the Frenet frame. Due to $\mathbf{b}'(s) \perp \mathbf{b}(s)$ we have $\mathbf{b}'(s) = \beta\cdot\mathbf{n}(s) + \gamma\cdot\boldsymbol{\tau}(s)$. The coefficients β and γ in this expansion can be found by means of the calculations similar to (3.5):

$$\begin{aligned}\beta(s) &= (\mathbf{n}(s)\,|\,\mathbf{b}'(s)) = (\mathbf{n}(s)\,|\,\mathbf{b}(s))' - (\mathbf{n}'(s)\,|\,\mathbf{b}(s)) = \\ &= -(-k(s)\cdot\boldsymbol{\tau}(s) + \varkappa(s)\cdot\mathbf{b}(s)\,|\,\mathbf{b}(s)) = -\varkappa(s).\\ \gamma(s) &= (\boldsymbol{\tau}(s)\,|\,\mathbf{b}'(s)) = (\boldsymbol{\tau}(s)\,|\,\mathbf{b}(s))' - (\boldsymbol{\tau}'(s)\,|\,\mathbf{b}(s)) = \\ &= -(k(s)\cdot\mathbf{n}(s)\,|\,\mathbf{b}(s)) = 0.\end{aligned}$$

Hence, for the expansion of the vector $\mathbf{b}'(s)$ in the Frenet frame we get

$$\mathbf{b}'(s) = -\varkappa(s)\cdot\mathbf{n}(s).\tag{3.7}$$

Let's gather the equations (3.2), (3.6), and (3.7) into a system:

$$\begin{cases}\boldsymbol{\tau}'(s) = k(s)\cdot\mathbf{n}(s),\\ \mathbf{n}'(s) = -k(s)\cdot\boldsymbol{\tau}(s) + \varkappa(s)\cdot\mathbf{b}(s),\\ \mathbf{b}'(s) = -\varkappa(s)\cdot\mathbf{n}(s).\end{cases}\tag{3.8}$$

The equations (3.8) relate the vectors $\boldsymbol{\tau}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$ and their derivatives with respect to s. These differential equations describe the dynamics of the Frenet frame. They are called the *Frenet equations*. The equations (3.8) should be complemented with the equation (3.1) which describes the dynamics of the point $\mathbf{r}(s)$ (the point to which the vectors of the Frenet frame are attached).

§ 4. The curvature center and the curvature radius of a spacial curve. The evolute and the evolvent of a curve.

In the case of a planar curve the vectors $\boldsymbol{\tau}(s)$ and $\mathbf{n}(s)$ lie in the same plane as the curve itself. Therefore, binormal vector (3.3) in this case coincides with the unit normal vector of the plane. Its derivative $\mathbf{b}'(s)$ is equal to zero. Hence, due to the third Frenet equation (3.7) we find that for a planar curve $\varkappa(s) \equiv 0$. The Frenet equations (3.8) then are reduced to

$$\begin{cases}\boldsymbol{\tau}'(s) = k(s)\cdot\mathbf{n}(s),\\ \mathbf{n}'(s) = -k(s)\cdot\boldsymbol{\tau}(s).\end{cases}\tag{4.1}$$

Let's consider the circle of the radius R with the center at the origin lying in the coordinate plane $x^3 = 0$. It is convenient to define this circle as follows:

$$\mathbf{r}(s) = \left\| \begin{matrix} R\,\cos(s/R) \\ R\,\sin(s/R) \end{matrix} \right\|, \tag{4.2}$$

here s is the natural parameter. Substituting (4.2) into (3.1) and then into (3.2), we find the unit tangent vector $\boldsymbol{\tau}(s)$ and the primary normal vector $\mathbf{n}(s)$:

$$\boldsymbol{\tau}(s) = \left\| \begin{matrix} -\sin(s/R) \\ \cos(s/R) \end{matrix} \right\|, \qquad \mathbf{n}(s) = \left\| \begin{matrix} -\cos(s/R) \\ -\sin(s/R) \end{matrix} \right\|. \tag{4.3}$$

Now, substituting (4.3) into the formula (4.1), we calculate the curvature of a circle $k(s) = 1/R = \text{const}$. The curvature k of a circle is constant, the inverse curvature $1/k$ coincides with its radius.

Let's make a step from the point $\mathbf{r}(s)$ on a circle to the distance $1/k$ in the direction of its primary normal vector $\mathbf{n}(s)$. It is easy to see that we come to the center of a circle. Let's make the same step for an arbitrary spacial curve. As a result of this step we come from the initial point $\mathbf{r}(s)$ on the curve to the point with the following radius-vector:

$$\boldsymbol{\rho}(s) = \mathbf{r}(s) + \frac{\mathbf{n}(s)}{k(s)}. \tag{4.4}$$

Certainly, this can be done only for that points of a curve, where $k(s) \neq 0$. The analogy with a circle induces the following terminology: the quantity $R(s) = 1/k(s)$ is called the *curvature radius*, the point with the radius-vector (4.4) is called the *curvature center* of a curve at the point $\mathbf{r}(s)$.

In the case of an arbitrary curve its curvature center is not a fixed point. When parameter s is varied, the curvature center of the curve moves in the space drawing another curve, which is called the *evolute* of the original curve. The formula (4.4) is a vectorial-parametric equation of the evolute. However, note that the natural parameter s of the original curve is not a natural parameter for its evolute.

Suppose that some spacial curve $\mathbf{r}(t)$ is given. A curve $\tilde{\mathbf{r}}(\tilde{s})$ whose evolute $\tilde{\boldsymbol{\rho}}(\tilde{s})$ coincides with the curve $\mathbf{r}(t)$ is called an *evolvent* of the curve $\mathbf{r}(t)$. The problem of constructing the evolute of a given curve is solved by the formula (4.4). The inverse problem of constructing an evolvent for a given curve appears to be more complicated. It is effectively solved only in the case of a planar curve.

Let $\mathbf{r}(s)$ be a vector-function defining some planar curve in natural parametrization and let $\tilde{\mathbf{r}}(\tilde{s})$ be the evolvent in its own natural parametrization. Two natural parameters s and $\tilde{s}$ are related to each other by some function φ in form of the relationship $\tilde{s} = \varphi(s)$. Let $\psi = \varphi^{-1}$ be the inverse function for φ, then $s = \psi(\tilde{s})$. Using the formula (4.4), now we obtain

$$\mathbf{r}(\psi(\tilde{s})) = \tilde{\mathbf{r}}(\tilde{s}) + \frac{\tilde{\mathbf{n}}(\tilde{s})}{\tilde{k}(\tilde{s})}. \tag{4.5}$$

Let's differentiate the relationship (4.5) with respect to $\tilde{s}$ and then let's apply the formula (3.1) and the Frenet equations written in form of (4.1):

$$\psi'(\tilde{s}) \cdot \boldsymbol{\tau}(\psi(\tilde{s})) = \frac{d}{d\tilde{s}} \left(\frac{1}{\tilde{k}(\tilde{s})} \right) \cdot \tilde{\mathbf{n}}(\tilde{s}).$$

Here $\boldsymbol{\tau}(\psi(\tilde{s}))$ and $\tilde{\mathbf{n}}(\tilde{s})$ both are unit vectors which are collinear due to the above relationship. Hence, we have the following two equalities:

$$\tilde{\mathbf{n}}(\tilde{s}) = \pm\boldsymbol{\tau}(\psi(\tilde{s})), \qquad\qquad \psi'(\tilde{s}) = \pm\frac{d}{d\tilde{s}}\left(\frac{1}{\tilde{k}(\tilde{s})}\right). \tag{4.6}$$

The second equality (4.6) can be integrated:

$$\frac{1}{\tilde{k}(\tilde{s})} = \pm(\psi(\tilde{s}) - C). \tag{4.7}$$

Here C is a constant of integration. Let's combine (4.7) with the first relationship (4.6) and substitute it into the formula (4.5):

$$\tilde{\mathbf{r}}(\tilde{s}) = \mathbf{r}(\psi(\tilde{s})) + (C - \psi(\tilde{s}))\cdot\boldsymbol{\tau}(\psi(\tilde{s})).$$

Then we substitute $\tilde{s} = \varphi(s)$ into the above formula and denote $\boldsymbol{\rho}(s) = \tilde{\mathbf{r}}(\varphi(s))$. As a result we obtain the following equality:

$$\boldsymbol{\rho}(s) = \mathbf{r}(s) + (C - s)\cdot\boldsymbol{\tau}(s). \tag{4.8}$$

The formula (4.8) is a parametric equation for the evolvent of a planar curve $\mathbf{r}(s)$. The entry of an arbitrary constant in the equation (4.8) means the evolvent is not unique. Each curve has the family of evolvents. This fact is valid for non-planar curves either. However, we should emphasize that the formula (4.8) cannot be applied to general spacial curves.

§ 5. Curves as trajectories of material points in mechanics.

The presentation of classical mechanics traditionally begins with considering the motion of material points. Saying *material point*, we understand any material object whose sizes are much smaller than its displacement in the space. The position of such an object can be characterized by its radius-vector in some Cartesian coordinate system, while its motion is described by a vector-function $\mathbf{r}(t)$. The curve $\mathbf{r}(t)$ is called the *trajectory* of a material point. Unlike to purely geometric curves, the trajectories of material points possess preferable parameter t, which is usually distinct from the natural parameter s. This preferable parameter is the time variable t.

The tangent vector of a trajectory, when computed in the time parametrization, is called the *velocity* of a material point:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}(t) = \begin{Vmatrix} v^1(t) \\ v^2(t) \\ v^3(t) \end{Vmatrix}. \tag{5.1}$$

The time derivative of the velocity vector is called the *acceleration vector*:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}(t) = \begin{Vmatrix} a^1(t) \\ a^2(t) \\ a^3(t) \end{Vmatrix}. \tag{5.2}$$

The motion of a material point in mechanics is described by Newton's second law:

$$m\,\mathbf{a} = \mathbf{F}(\mathbf{r}, \mathbf{v}). \tag{5.3}$$

Here m is the mass of a material point. This is a constant characterizing the amount of matter enclosed in this material object. The vector $\mathbf{F}$ is the *force vector*. By means of the force vector in mechanics one describes the action of ambient objects (which are sometimes very far apart) upon the material point under consideration. The magnitude of this action usually depends on the position of a point relative to the ambient objects, but sometimes it can also depend on the velocity of the point itself. Newton's second law in form of (5.3) shows that the external action immediately affects the acceleration of a material point, but neither the velocity nor the coordinates of a point.

Let $s = s(t)$ be the natural parameter on the trajectory of a material point expressed through the time variable. Then the formula (2.5) yields

$$\dot{s}(t) = |\mathbf{v}(t)| = v(t). \tag{5.4}$$

Through $v(t)$ in (5.4) we denote the modulus of the velocity vector.

Let's consider a trajectory of a material point in natural parametrization: $\mathbf{r} = \mathbf{r}(s)$. Then for the velocity vector (5.1) and for the acceleration vector (5.2) we get the following expressions:

$$\begin{aligned}
&\mathbf{v}(t) = \dot{s}(t)\cdot\boldsymbol{\tau}(s(t)),\\
&\mathbf{a}(t) = \ddot{s}(t)\cdot\boldsymbol{\tau}(s(t)) + (\dot{s}(t))^2\cdot\boldsymbol{\tau}'(s(t)).
\end{aligned}$$

Taking into account the formula (5.4) and the first Frenet equation, these expressions can be rewritten as

$$\begin{aligned}
&\mathbf{v}(t) = v(t)\cdot\boldsymbol{\tau}(s(t)),\\
&\mathbf{a}(t) = \dot{v}(t)\cdot\boldsymbol{\tau}(s(t)) + \bigl(k(s(t))\,v(t)^2\bigr)\cdot\mathbf{n}(s(t)).
\end{aligned} \tag{5.5}$$

The second formula (5.5) determines the expansion of the acceleration vector into two components. The first component is tangent to the trajectory, it is called the *tangential acceleration*. The second component is perpendicular to the trajectory and directed toward the curvature center. It is called the *centripetal acceleration*. It is important to note that the centripetal acceleration is determined by the modulus of the velocity and by the geometry of the trajectory (by its curvature).

CHAPTER II

ELEMENTS OF VECTORIAL AND TENSORIAL ANALYSIS.

§ 1. Vectorial and tensorial fields in the space.

Let again $\mathbb{E}$ be a three-dimensional Euclidean point space. We say that in $\mathbb{E}$ a *vectorial field* is given if at each point of the space $\mathbb{E}$ some vector attached to this point is given. Let's choose some Cartesian coordinate system in $\mathbb{E}$; in general, this system is skew-angular. Then we can define the points of the space by their coordinates x^1, x^2, x^3, and, simultaneously, we get the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for expanding the vectors attached to these points. In this case we can present any vector field $\mathbf{F}$ by three numeric functions

$$\mathbf{F} = \begin{Vmatrix} F^1(\mathbf{x}) \\ F^2(\mathbf{x}) \\ F^3(\mathbf{x}) \end{Vmatrix}, \tag{1.1}$$

where $\mathbf{x} = (x^1, x^2, x^3)$ are the components of the radius-vector of an arbitrary point of the space $\mathbb{E}$. Writing $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x^1, x^2, x^3)$, we make all formulas more compact.

The vectorial nature of the field $\mathbf{F}$ reveals when we replace one coordinate system by another. Let (1.1) be the coordinates of a vector field in some coordinate system O, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and let $\tilde{O}$, $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$ be some other coordinate system. The transformation rule for the components of a vectorial field under a change of a Cartesian coordinate system is written as follows:

$$\begin{aligned} F^i(\mathbf{x}) &= \sum_{j=1}^{3} S^i_j\ \tilde{F}^j(\tilde{\mathbf{x}}),\\ x^i &= \sum_{j=1}^{3} S^i_j\,\tilde{x}^j + a^i. \end{aligned} \tag{1.2}$$

Here S^i_j are the components of the transition matrix relating the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with the new basis $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$, while a^1, a^2, a^3 are the components of the vector $\overrightarrow{O\tilde{O}}$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

The formula (1.2) combines the transformation rule for the components of a vector under a change of a basis and the transformation rule for the coordinates of a point under a change of a Cartesian coordinate system (see [1]). The arguments $\mathbf{x}$ and $\tilde{\mathbf{x}}$ beside the vector components F^i and $\tilde{F}^i$ in (1.2) is an important novelty as compared to [1]. It is due to the fact that here we deal with vector fields, not with separate vectors.

Not only vectors can be associated with the points of the space $\mathbb{E}$. In linear algebra along with vectors one considers covectors, linear operators, bilinear forms and quadratic forms. Associating some covector with each point of $\mathbb{E}$, we get a *covector* field. If we associate some linear operator with each point of the space, we get an *operator field*. An finally, associating a bilinear (quadratic) form with each point of $\mathbb{E}$, we obtain a *field of bilinear (quadratic) forms*. Any choice of a Cartesian coordinate system O, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ assumes the choice of a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, while the basis defines the numeric representations for all of the above objects: for a covector this is the list of its components, for linear operators, bilinear and quadratic forms these are their matrices. Therefore defining a covector field $\mathbf{F}$ is equivalent to defining three functions $F_1(\mathbf{x})$, $F_2(\mathbf{x})$, $F_3(\mathbf{x})$ that transform according to the following rule under a change of a coordinate system:

$$\begin{aligned} F_i(\mathbf{x}) &= \sum_{j=1}^{3} T^j_i\;\tilde{F}_j(\tilde{\mathbf{x}}),\\ x^i &= \sum_{j=1}^{3} S^i_j\,\tilde{x}^j + a^i. \end{aligned} \tag{1.3}$$

In the case of operator field $\mathbf{F}$ the transformation formula for the components of its matrix under a change of a coordinate system has the following form:

$$\begin{aligned} F^i_j(\mathbf{x}) &= \sum_{p=1}^{3}\sum_{q=1}^{3} S^i_p\,T^q_j\;\tilde{F}^p_q(\tilde{\mathbf{x}}),\\ x^i &= \sum_{p=1}^{3} S^i_p\,\tilde{x}^p + a^i. \end{aligned} \tag{1.4}$$

For a field of bilinear (quadratic) forms $\mathbf{F}$ the transformation rule for its components under a change of Cartesian coordinates looks like

$$\begin{aligned} F_{ij}(\mathbf{x}) &= \sum_{p=1}^{3}\sum_{q=1}^{3} T^p_i\,T^q_j\;\tilde{F}_{p\,q}(\tilde{\mathbf{x}}),\\ x^i &= \sum_{p=1}^{3} S^i_p\,\tilde{x}^p + a^i. \end{aligned} \tag{1.5}$$

Each of the relationships (1.2), (1.3), (1.4), and (1.5) consists of two formulas. The first formula relates the components of a field, which are the functions of two different sets of arguments $\mathbf{x} = (x^1, x^2, x^3)$ and $\tilde{\mathbf{x}} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$. The second formula establishes the functional dependence of these two sets of arguments.

The first formulas in (1.2), (1.3), and (1.4) are different. However, one can see some regular pattern in them. The number of summation signs and the number of summation indices in their right hand sides are determined by the number of indices in the components of a field $\mathbf{F}$. The total number of transition matrices used in the right hand sides of these formulas is also determined by the number of indices in the components of $\mathbf{F}$. Thus, each upper index of $\mathbf{F}$ implies the usage of the transition matrix S, while each lower index of $\mathbf{F}$ means that the inverse matrix $T = S^{-1}$ is used.

The number of indices of the field $\mathbf{F}$ in the above examples doesn't exceed two. However, the regular pattern detected in the transformation rules for the components of $\mathbf{F}$ can be generalized for the case of an arbitrary number of indices:

$$F^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1} \ldots S^{i_r}_{p_r}\, T^{q_1}_{j_1} \ldots T^{q_s}_{j_s}\, \tilde{F}^{p_1\ldots p_r}_{q_1\ldots q_s} \tag{1.6}$$

The formula (1.6) comprises the multiple summation with respect to $(r+s)$ indices $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ each of which runs from 1 to 3.

DEFINITION 1.1. A tensor of the type (r, s) is a geometric object $\mathbf{F}$ whose components in each basis are enumerated by $(r+s)$ indices and obey the transformation rule (1.6) under a change of basis.

Lower indices in the components of a tensor are called *covariant indices*, upper indices are called *contravariant indices* respectively. Generalizing the concept of a vector field, we can attach some tensor of the type (r, s), to each point of the space. As a result we get the concept of a *tensor field*. This concept is convenient because it describes in the unified way any vectorial and covectorial fields, operator fields, and arbitrary fields of bilinear (quadratic) forms. Vectorial fields are fields of the type $(1, 0)$, covectorial fields have the type $(0, 1)$, operator fields are of the type $(1, 1)$, and finally, any field of bilinear (quadratic) forms are of the type $(0, 2)$. Tensor fields of some other types are also meaningful. In Chapter IV we consider the curvature field with four indices.

Passing from separate tensors to tensor fields, we acquire the arguments in formula (1.6). Now this formula should be written as the couple of two relationships similar to (1.2), (1.3), (1.4), or (1.5):

$$\begin{aligned} &F^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}(\mathbf{x}) = \sum_{\substack{p_1\ldots\, p_r\\ q_1\ldots\, q_s}} S^{i_1}_{p_1} \ldots S^{i_r}_{p_r}\, T^{q_1}_{j_1} \ldots T^{q_s}_{j_s}\, \tilde{F}^{p_1\ldots\, p_r}_{q_1\ldots\, q_s}(\tilde{\mathbf{x}}),\\ &x^i = \sum_{j=1}^{3} S^i_j\, \tilde{x}^j + a^i. \end{aligned} \tag{1.7}$$

The formula (1.7) expresses the transformation rule for the components of a tensorial field of the type (r, s) under a change of Cartesian coordinates.

The most simple type of tensorial fields is the type $(0, 0)$. Such fields are called *scalar fields*. Their components have no indices at all, i. e. they are numeric functions in the space $\mathbb{E}$.

§ 2. Tensor product and contraction.

Let's consider two covectorial fields $\mathbf{a}$ and $\mathbf{b}$. In some Cartesian coordinate system they are given by their components $a_i(\mathbf{x})$ and $b_j(\mathbf{x})$. These are two sets of functions with three functions in each set. Let's form a new set of nine functions by multiplying the functions of initial sets:

$$c_{ij}(\mathbf{x}) = a_i(\mathbf{x})\, b_j(\mathbf{x}). \tag{2.1}$$

Applying the formula (1.3) we can express the right hand side of (2.1) through the components of the fields $\mathbf{a}$ and $\mathbf{b}$ in the other coordinate system:

$$c_{ij}(\mathbf{x}) = \left(\sum_{p=1}^{3} T^p_i\, \tilde{a}_p\right) \left(\sum_{q=1}^{3} T^q_j\, \tilde{b}_q\right) = \sum_{p=1}^{3}\sum_{q=1}^{3} T^p_i\, T^q_j\, (\tilde{a}_p\, \tilde{b}_q).$$

If we denote by $\tilde{c}_{pq}(\tilde{\mathbf{x}})$ the product of $\tilde{a}_i(\tilde{\mathbf{x}})$ and $\tilde{b}_j(\tilde{\mathbf{x}})$, then we find that the quantities $c_{ij}(\mathbf{x})$ and $\tilde{c}_{pq}(\tilde{\mathbf{x}})$ are related by the formula (1.5). This means that taking two covectorial fields one can compose a field of bilinear forms by multiplying the components of these two covectorial fields in an arbitrary Cartesian coordinate system. This operation is called the *tensor product* of the fields $\mathbf{a}$ and $\mathbf{b}$. Its result is denoted as $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$.

The above trick of multiplying components can be applied to an arbitrary pair of tensor fields. Suppose we have a tensorial field $\mathbf{A}$ of the type (r, s) and another tensorial field $\mathbf{B}$ of the type (m, n). Denote

$$C^{i_1\dots\, i_r i_{r+1}\dots\, i_{r+m}}_{j_1\dots\, j_s j_{s+1}\dots\, j_{s+n}}(\mathbf{x}) = A^{i_1\dots\, i_r}_{j_1\dots\, j_s}(\mathbf{x})\ B^{i_{r+1}\dots\, i_{r+m}}_{j_{s+1}\dots\, j_{s+n}}(\mathbf{x}). \tag{2.2}$$

Definition 2.1. The tensor field $\mathbf{C}$ of the type $(r{+}m, s{+}n)$ whose components are determined by the formula (2.2) is called the *tensor product* of the fields $\mathbf{A}$ and $\mathbf{B}$. It is denoted $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$.

This definition should be checked for correctness. We should make sure that the components of the field $\mathbf{C}$ are transformed according to the rule (1.7) when we pass from one Cartesian coordinate system to another. The transformation rule (1.7), when applied to the fields $\mathbf{A}$ and $\mathbf{B}$, yields

$$\begin{aligned}
&A^{i_1\dots\, i_r}_{j_1\dots\, j_s} = \sum_{p..q} S^{i_1}_{p_1}\dots S^{i_r}_{p_r}\ T^{q_1}_{j_1}\dots T^{q_s}_{j_s}\ \tilde{A}^{p_1\dots\, p_r}_{q_1\dots\, q_s},\\
&B^{i_{r+1}\dots\, i_{r+m}}_{j_{s+1}\dots\, j_{s+n}} = \sum_{p..q} S^{i_{r+1}}_{p_{r+1}}\dots S^{i_{r+m}}_{p_{r+m}}\ T^{q_{s+1}}_{j_{s+1}}\dots T^{q_{s+n}}_{j_{s+n}}\ \tilde{B}^{p_{r+1}\dots\, p_{r+m}}_{q_{s+1}\dots\, q_{s+n}}.
\end{aligned}$$

The summation in right hand sides of this formulas is carried out with respect to each double index which enters the formula twice — once as an upper index and once as a lower index. Multiplying these two formulas, we get exactly the transformation rule (1.7) for the components of $\mathbf{C}$.

Theorem 2.1. *The operation of tensor product is associative, this means that* $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$.

Proof. Let $\mathbf{A}$ be a tensor of the type (r, s), let $\mathbf{B}$ be a tensor of the type (m, n), and let $\mathbf{C}$ be a tensor of the type (p, q). Then one can write the following obvious numeric equality for their components:

$$\begin{aligned}
&\left(A^{i_1\dots\, i_r}_{j_1\dots\, j_s}\ B^{i_{r+1}\dots\, i_{r+m}}_{j_{s+1}\dots\, j_{s+n}}\right) C^{\,i_{r+m+1}\dots\, i_{r+m+p}}_{j_{s+n+1}\dots\, j_{s+n+q}} =\\
&\quad = A^{i_1\dots\, i_r}_{j_1\dots\, j_s}\ \left(B^{i_{r+1}\dots\, i_{r+m}}_{j_{s+1}\dots\, j_{s+n}}\ C^{\,i_{r+m+1}\dots\, i_{r+m+p}}_{j_{s+n+1}\dots\, j_{s+n+q}}\right).
\end{aligned} \tag{2.3}$$

As we see in (2.3), the associativity of the tensor product follows from the associativity of the multiplication of numbers. □

The tensor product is not commutative. One can easily construct an example illustrating this fact. Let's consider two covectorial fields $\mathbf{a}$ and $\mathbf{b}$ with the following components in some coordinate system: $\mathbf{a} = (1, 0, 0)$ and $\mathbf{b} = (0, 1, 0)$. Denote $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$ and $\mathbf{d} = \mathbf{b} \otimes \mathbf{a}$. Then for c_{12} and d_{12} with the use of the formula (2.2) we derive: $c_{12} = 1$ and $d_{12} = 0$. Hence, $\mathbf{c} \neq \mathbf{d}$ and $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a}$.

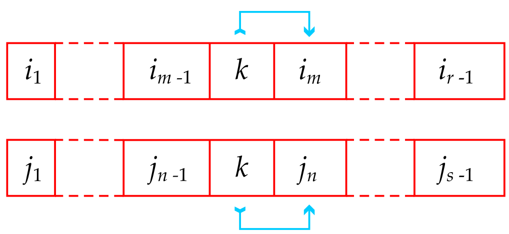
Let's consider an operator field $\mathbf{F}$. Its components $F^i_j(\mathbf{x})$ are the components of the operator $\mathbf{F}(\mathbf{x})$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. It is known that the trace of the matrix $F^i_j(\mathbf{x})$ is a scalar invariant of the operator $\mathbf{F}(\mathbf{x})$ (see [1]). Therefore, the formula

$$f(\mathbf{x}) = \operatorname{tr}\mathbf{F}(\mathbf{x}) = \sum_{i=1}^{3} F^i_i(\mathbf{x}) \tag{2.4}$$

determines a scalar field $f(\mathbf{x})$ in the space $\mathbb{E}$. The sum similar to (2.4) can be written for an arbitrary tensorial field $\mathbf{F}$ with at least one upper index and at least one lower index in its components:

$$H^{i_1\ldots\,i_{r-1}}_{j_1\ldots\,j_{s-1}}(\mathbf{x}) = \sum_{k=1}^{3} F^{i_1\ldots\,i_{m-1}\,k\,i_m\ldots\,i_{r-1}}_{j_1\ldots\,j_{n-1}\,k\,j_n\ldots\,j_{s-1}}(\mathbf{x}). \tag{2.5}$$

In the formula (2.5) the summation index k is placed to m-th upper position and to n-th lower position. The succeeding indices i_m, ... i_{r-1} and j_n, ... j_{s-1} in writing the components of the field $\mathbf{F}$ are shifted one position to the right as compared to their positions in left hand side of the equality (2.5):



Definition 2.2. The tensor field $\mathbf{H}$ whose components are calculated according to the formula (2.5) from the components of the tensor field $\mathbf{F}$ is called the *contraction* of the field $\mathbf{F}$ with respect to m-th and n-th indices.

Like the definition 2.1, this definition should be tested for correctness. Let's verify that the components of the field $\mathbf{H}$ are transformed according to the formula (1.7). For this purpose we write the transformation rule (1.7) applied to the components of the field $\mathbf{F}$ in right hand side of the formula (2.5):

$$\begin{gathered}
F^{i_1\ldots\,i_{m-1}\,k\,i_m\ldots\,i_{r-1}}_{j_1\ldots\,j_{n-1}\,k\,j_n\ldots\,j_{s-1}} = \sum_{\substack{\alpha\,p_1\ldots p_{r-1}\\ \beta\,q_1\ldots q_{s-1}}} S^{i_1}_{p_1}\ldots\,S^{i_{m-1}}_{p_{m-1}}\,S^k_\alpha\,S^{i_m}_{p_m}\,\ldots\,S^{i_{r-1}}_{p_{r-1}}\,\times\\
\times\,T^{q_1}_{j_1}\,\ldots\,T^{q_{n-1}}_{j_{n-1}}\,T^\beta_k\,T^{q_n}_{j_n}\,\ldots\,T^{q_{s-1}}_{j_{s-1}}\;\tilde F^{p_1\ldots\,p_{m-1}\,\alpha\,p_m\ldots\,p_{r-1}}_{q_1\ldots\,q_{n-1}\,\beta\,q_n\ldots\,q_{s-1}}.
\end{gathered}$$

In order to derive this formula from (1.7) we substitute the index k into the m-th and n-th positions, then we shift all succeeding indices one position to the right. In order to have more similarity of left and right hand sides of this formula we shift summation indices as well. It is clear that such redesignation of summation indices does not change the value of the sum.

Now in order to complete the contraction procedure we should produce the summation with respect to the index k. In the right hand side of the formula the sum over k can be calculated explicitly due to the formula

$$\sum_{k=1}^{3} S^k_\alpha\,T^\beta_k = \delta^\beta_\alpha, \tag{2.6}$$

which means $T = S^{-1}$. Due to (2.6) upon calculating the sum over k one can calculate the sums over β and α. Therein we take into account that

$$\sum_{\alpha=1}^{3} \tilde{F}^{p_1\ldots\,p_{m-1}\,\alpha\,p_m\ldots\,p_{r-1}}_{q_1\ldots\,q_{n-1}\,\alpha\,q_n\ldots\,q_{s-1}} = \tilde{H}^{p_1\ldots\,p_{r-1}}_{q_1\ldots\,q_{s-1}}.$$

As a result we get the equality

$$H^{i_1\ldots\,i_{r-1}}_{j_1\ldots\,j_{s-1}} = \sum_{\substack{p_1\ldots p_{r-1}\\ q_1\ldots q_{s-1}}} S^{i_1}_{p_1}\ldots S^{i_{r-1}}_{p_{r-1}}\;T^{q_1}_{j_1}\ldots T^{q_{s-1}}_{j_{s-1}}\;\tilde{H}^{p_1\ldots\,p_{r-1}}_{q_1\ldots\,q_{s-1}},$$

which exactly coincides with the transformation rule (1.7) written with respect to components of the field $\mathbf{H}$. The correctness of the definition 2.2 is proved.

The operation of contraction introduced by the definition 2.2 implies that the positions of two indices are specified. One of these indices should be an upper index, the other index should be a lower index. The letter C is used as a contraction sign. The formula (2.5) then is abbreviated as follows:

$$\mathbf{H} = C_{m,n}(\mathbf{F}) = C(\mathbf{F}).$$

The numbers m and n are often omitted since they are usually known from the context.

A tensorial field of the type $(1,1)$ can be contracted in the unique way. For a tensorial field $\mathbf{F}$ of the type $(2,2)$ we have two ways of contracting. As a result of these two contractions, in general, we obtain two different tensorial fields of the type $(1,1)$. These tensorial fields can be contracted again. As a result we obtain the *complete contractions* of the field $\mathbf{F}$, they are scalar fields. A field of the type $(2,2)$ can have two complete contractions. In general case a field of the type (n,n) has $n!$ complete contractions.

The operations of tensor product and contraction often arise in a natural way without any special intension. For example, suppose that we are given a vector field $\mathbf{v}$ and a covector field $\mathbf{w}$ in the space $\mathbb{E}$. This means that at each point we have a vector and a covector attached to this point. By calculating the scalar products of these vectors and covectors we get a scalar field $f = \langle \mathbf{w}\,|\,\mathbf{v}\rangle$. In coordinate form such a scalar field is calculated by means of the formula

$$f = \sum_{k=1}^{3} w_i\,v^i. \tag{2.7}$$

From the formula (2.7), it is clear that $f = C(\mathbf{w}\otimes\mathbf{v})$. The scalar product $f = \langle \mathbf{w}\,|\,\mathbf{v}\rangle$ is the contraction of the tensor product of the fields $\mathbf{w}$ and $\mathbf{v}$. In a similar way, if an operator field $\mathbf{F}$ and a vector field $\mathbf{v}$ are given, then applying $\mathbf{F}$ to $\mathbf{v}$ we get another vector field $\mathbf{u} = \mathbf{F}\,\mathbf{v}$, where

$$u^i = \sum_{j=1}^{3} F^i_j\,v^j.$$

In this case we can write: $\mathbf{u} = C(\mathbf{F}\otimes\mathbf{v})$; although this writing cannot be uniquely interpreted. Apart from $\mathbf{u} = \mathbf{F}\,\mathbf{v}$, it can mean the product of $\mathbf{v}$ by the trace of the operator field $\mathbf{F}$.

§ 3. The algebra of tensor fields.

Let $\mathbf{v}$ and $\mathbf{w}$ be two vectorial fields. Then at each point of the space $\mathbb{E}$ we have two vectors $\mathbf{v}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$. We can add them. As a result we get a new vector field $\mathbf{u} = \mathbf{v} + \mathbf{w}$. In a similar way one can define the addition of tensor fields. Let $\mathbf{A}$ and $\mathbf{B}$ be two tensor fields of the type (r, s). Let's consider the sum of their components in some Cartesian coordinate system:

$$C^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}. \tag{3.1}$$

Definition 3.1. The tensor field $\mathbf{C}$ of the type (r, s) whose components are calculated according to the formula (3.1) is called the *sum* of the fields $\mathbf{A}$ and $\mathbf{B}$ of the type (r, s).

One can easily check up the transformation rule (1.7) for the components of the field $\mathbf{C}$. It is sufficient to write this rule (1.7) for the components of $\mathbf{A}$ and $\mathbf{B}$ then add these two formulas. Therefore, the definition 3.1 is consistent.

The sum of tensor fields is commutative and associative. This fact follows from the commutativity and associativity of the addition of numbers due to the following obvious relationships:

$$\begin{aligned}
&A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s},\\
&\bigl(A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\bigr) + C^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + \bigl(B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + C^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\bigr).
\end{aligned}$$

Let's denote by $T_{(r,s)}$ the set of tensor fields of the type (r, s). The tensor multiplication introduced by the definition 2.1 is the following binary operation:

$$T_{(r,\,s)} \times T_{(m,\,n)} \to T_{(r+m,\,s+n)}. \tag{3.2}$$

The operations of tensor addition and tensor multiplication (3.2) are related to each other by the distributivity laws:

$$\begin{aligned}
&(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C},\\
&\mathbf{C} \otimes (\mathbf{A} + \mathbf{B}) = \mathbf{C} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{B}.
\end{aligned} \tag{3.3}$$

The distributivity laws (3.3) follow from the distributivity of the multiplication of numbers. Their proof is given by the following obvious formulas:

$$\begin{aligned}
&\bigl(A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\bigr)\, C^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}} =\\
&\qquad = A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\, C^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}} + B^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\, C^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}},\\
&C^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\left(A^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}} + B^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}}\right) =\\
&\qquad = C^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\, A^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}} + C^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}\, B^{i_{r+1}\ldots\, i_{r+m}}_{j_{s+1}\ldots\, j_{s+n}}.
\end{aligned}$$

Due to (3.2) the set of scalar fields $K = T_{(0,0)}$ (which is simply the set of numeric functions) is closed with respect to tensor multiplication $\otimes$, which coincides here with the regular multiplication of numeric functions. The set K is a *commutative ring* (see [3]) *with the unity*. The constant function equal to 1 at each point of the space $\mathbb{E}$ plays the role of the unit element in this ring.

Let's set $m = n = 0$ in the formula (3.2). In this case it describes the multiplication of tensor fields from $T_{(r,s)}$ by numeric functions from the ring K. The tensor product of a field $\mathbf{A}$ and a scalar filed $\xi \in K$ is commutative: $\mathbf{A} \otimes \xi = \xi \otimes \mathbf{A}$. Therefore, the multiplication of tensor fields by numeric functions is denoted by standard sign of multiplication: $\xi \otimes \mathbf{A} = \xi \cdot \mathbf{A}$. The operation of addition and the operation of multiplication by scalar fields in the set $T_{(r,s)}$ possess the following properties:

(1) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$;
(2) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$;
(3) there exists a field $\mathbf{0} \in T_{(r,s)}$ such that $\mathbf{A} + \mathbf{0} = \mathbf{A}$ for an arbitrary tensor field $\mathbf{A} \in T_{(r,s)}$;
(4) for any tensor field $\mathbf{A} \in T_{(r,s)}$ there exists an opposite field $\mathbf{A}'$ such that $\mathbf{A} + \mathbf{A}' = \mathbf{0}$;
(5) $\xi \cdot (\mathbf{A} + \mathbf{B}) = \xi \cdot \mathbf{A} + \xi \cdot \mathbf{B}$ for any function ξ from the ring K and for any two fields $\mathbf{A}, \mathbf{B} \in T_{(r,s)}$;
(6) $(\xi + \zeta) \cdot \mathbf{A} = \xi \cdot \mathbf{A} + \zeta \cdot \mathbf{A}$ for any tensor field $\mathbf{A} \in T_{(r,s)}$ and for any two functions $\xi, \zeta \in K$;
(7) $(\xi\,\zeta) \cdot \mathbf{A} = \xi \cdot (\zeta \cdot \mathbf{A})$ for any tensor field $\mathbf{A} \in T_{(r,s)}$ and for any two functions $\xi, \zeta \in K$;
(8) $1 \cdot \mathbf{A} = \mathbf{A}$ for any field $\mathbf{A} \in T_{(r,s)}$.

The tensor field with identically zero components plays the role of zero element in the property (3). The field $\mathbf{A}'$ in the property (4) is defined as a field whose components are obtained from the components of $\mathbf{A}$ by changing the sign.

The properties (1)-(8) listed above almost literally coincide with the axioms of a linear vector space (see [1]). The only discrepancy is that the set of functions K is a ring, not a numeric field as it should be in the case of a linear vector space. The sets defined by the axioms (1)-(8) for some ring K are called *modules over the ring* K or K*-modules*. Thus, each of the sets $T_{(r,s)}$ is a module over the ring of scalar functions $K = T_{(0,0)}$.

The ring $K = T_{(0,0)}$ comprises the subset of constant functions which is naturally identified with the set of real numbers $\mathbb{R}$. Therefore the set of tensor fields $T_{(r,s)}$ in the space $\mathbb{E}$ is a linear vector space over the field of real numbers $\mathbb{R}$.

If $r \geqslant 1$ and $s \geqslant 1$, then in the set $T_{(r,s)}$ the operation of contraction with respect to various pairs of indices are defined. These operations are linear, i. e. the following relationships are fulfilled:

$$\begin{aligned} &C(\mathbf{A} + \mathbf{B}) = C(\mathbf{A}) + C(\mathbf{B}),\\ &C(\xi \cdot \mathbf{A}) = \xi \cdot C(\mathbf{A}). \end{aligned} \tag{3.4}$$

The relationships (3.4) are proved by direct calculations in coordinates. For the field $\mathbf{C} = \mathbf{A} + \mathbf{B}$ from (2.5) we derive

$$\begin{aligned} &H^{i_1 \dots i_{r-1}}_{j_1 \dots j_{s-1}} = \sum_{k=1}^{3} C^{i_1 \dots i_{m-1}\, k\, i_m \dots i_{r-1}}_{j_1 \dots j_{n-1}\, k\, j_n \dots j_{s-1}} =\\ &= \sum_{k=1}^{3} A^{i_1 \dots i_{m-1}\, k\, i_m \dots i_{r-1}}_{j_1 \dots j_{n-1}\, k\, j_n \dots j_{s-1}} + \sum_{k=1}^{3} B^{i_1 \dots i_{m-1}\, k\, i_m \dots i_{r-1}}_{j_1 \dots j_{n-1}\, k\, j_n \dots j_{s-1}}. \end{aligned}$$

This equality proves the first relationship (3.4). In order to prove the second one we take $\mathbf{C} = \xi \cdot \mathbf{A}$. Then the second relationship (3.4) is derived as a result of the following calculations:

$$H^{i_1\ldots\,i_{r-1}}_{j_1\ldots\,j_{s-1}} = \sum_{k=1}^{3} C^{\,i_1\ldots\,i_{m-1}\,k\,i_m\ldots\,i_{r-1}}_{\,j_1\ldots\,j_{n-1}\,k\,j_n\ldots\,j_{s-1}} =$$

$$= \sum_{k=1}^{3} \xi\, A^{i_1\ldots\,i_{m-1}\,k\,i_m\ldots\,i_{r-1}}_{j_1\ldots\,j_{n-1}\,k\,j_n\ldots\,j_{s-1}} = \xi \sum_{k=1}^{3} A^{i_1\ldots\,i_{m-1}\,k\,i_m\ldots\,i_{r-1}}_{j_1\ldots\,j_{n-1}\,k\,j_n\ldots\,j_{s-1}}.$$

The tensor product of two tensors from $T_{(r,s)}$ belongs to $T_{(r,s)}$ only if $r = s = 0$ (see formula (3.2)). In all other cases one cannot perform the tensor multiplication staying within one K-module $T_{(r,s)}$. In order to avoid this restriction the following direct sum is usually considered:

$$T = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T_{(r,s)}. \tag{3.5}$$

The set (3.5) consists of finite formal sums $\mathbf{A}^{(1)} + \ldots + \mathbf{A}^{(k)}$, where each summand belongs to some of the K-modules $T_{(r,s)}$. The operation of tensor product is extended to the K-module T by means of the formula:

$$(\mathbf{A}^{(1)} + \ldots + \mathbf{A}^{(k)}) \otimes (\mathbf{A}^{(1)} + \ldots + \mathbf{A}^{(q)}) = \sum_{i=1}^{k} \sum_{j=1}^{q} \mathbf{A}^{(i)} \otimes \mathbf{A}^{(j)}.$$

This extension of the operation of tensor product is a bilinear binary operation in the set T. It possesses the following additional properties:

(9) $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$;
(10) $(\xi \cdot \mathbf{A}) \otimes \mathbf{C} = \xi \cdot (\mathbf{A} \otimes \mathbf{C})$;
(11) $\mathbf{C} \otimes (\mathbf{A} + \mathbf{B}) = \mathbf{C} \otimes \mathbf{A} + \mathbf{C} \otimes \mathbf{B}$;
(12) $\mathbf{C} \otimes (\xi \cdot \mathbf{B}) = \xi \cdot (\mathbf{C} \otimes \mathbf{B})$.

These properties of the operation of tensor product in T are easily derived from (3.3). Note that a K-module equipped with an additional bilinear binary operation of multiplication is called an *algebra over the ring* K or a K*-algebra*. Therefore the set T is called the *algebra of tensor fields*.

The algebra T is a direct sum of separate K-modules $T_{(r,s)}$ in (3.5). The operation of multiplication is concordant with this expansion into a direct sum; this fact is expressed by the relationship (3.2). Such structures in algebras are called *gradings*, while algebras with gradings are called *graded algebras*.

§ 4. Symmetrization and alternation.

Let $\mathbf{A}$ be a tensor filed of the type (r, s) and let $r \geqslant 2$. The number of upper indices in the components of the field $\mathbf{A}$ is greater than two. Therefore, we can perform the permutation of some pair of them. Let's denote

$$B^{i_1\,\ldots\,i_m\ldots\,i_n\ldots\,i_r}_{j_1\,\ldots\ldots\ldots\ldots\ldots\ldots\,j_s} = A^{i_1\,\ldots\,i_n\ldots\,i_m\ldots\,i_r}_{j_1\,\ldots\ldots\ldots\ldots\ldots\ldots\,j_s}. \tag{4.1}$$

The quantities $B^{i_1\,\ldots\,i_r}_{j_1\,\ldots\,j_s}$ in (4.1) are produced from the components of the tensor field $\mathbf{A}$ by the transposition of the pair of upper indices i_m and i_n.

THEOREM 4.1. *The quantities $B^{i_1\,\ldots\,i_r}_{j_1\,\ldots\,j_s}$ produced from the components of a tensor field* $\mathbf{A}$ *by the transposition of any pair of upper indices define another tensor field* $\mathbf{B}$ *of the same type as the original field* $\mathbf{A}$.

PROOF. In order to prove the theorem let's check up that the quantities (4.1) obey the transformation rule (1.7) under a change of a coordinate system:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}}S^{i_1}_{p_1}\ldots\,S^{i_n}_{p_m}\ldots\,S^{i_m}_{p_n}\ldots\,S^{i_r}_{p_r}\;T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\;\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}.$$

Let's rename the summation indices p_m and p_n in this formula: let's denote p_m by p_n and vice versa. As a result the S matrices will be arranged in the order of increasing numbers of their upper and lower indices. However, the indices p_m and p_n in $\tilde{A}^{p_1\,\ldots\,p_r}_{q_1\ldots\,q_s}$ will exchange their positions. It is clear that the procedure of renaming summation indices does not change the value of the sum:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}}S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\;T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\;\tilde{A}^{p_1\ldots\,p_n\ldots\,p_m\ldots\,p_r}_{q_1\ldots\,q_s}.$$

Due to the equality $\tilde{B}^{p_1\,\ldots\,p_r}_{q_1\ldots\,q_s}=\tilde{A}^{p_1\,\ldots\,p_n\ldots\,p_m\ldots\,p_r}_{q_1\,\ldots\ldots\ldots\ldots\,q_s}$ the above formula is exactly the transformation rule (1.7) written for the quantities (4.1). Hence, they define a tensor field $\mathbf{B}$. The theorem is proved. $\square$

There is a similar theorem for transpositions of lower indices. Let again $\mathbf{A}$ be a tensor field of the type (r,s) and let $s\geqslant 2$. Denote

$$B^{i_1\,\ldots\ldots\ldots\ldots\,i_r}_{j_1\,\ldots\,j_m\ldots\,j_n\ldots\,j_s}=A^{i_1\,\ldots\ldots\ldots\ldots\,i_r}_{j_1\,\ldots\,j_n\,\ldots\,j_m\ldots\,j_s}.\tag{4.2}$$

THEOREM 4.2. *The quantities $B^{i_1\,\ldots\,i_r}_{j_1\,\ldots\,j_s}$ produced from the components of a tensor field* $\mathbf{A}$ *by the transposition of any pair of lower indices define another tensor field* $\mathbf{B}$ *of the same type as the original field* $\mathbf{A}$.

The proof of the theorem 4.2 is completely analogous to the proof of the theorem 4.1. Therefore we do not give it here. Note that one cannot transpose an upper index and a lower index. The set of quantities obtained by such a transposition does not obey the transformation rule (1.7).

Combining various pairwise transpositions of indices (4.1) and (4.2) we can get any transposition from the symmetric group $\mathfrak{S}_r$ in upper indices and any transposition from the symmetric group $\mathfrak{S}_s$ in lower indices. This is a well-known fact from the algebra (see [3]). Thus the theorems 4.1 and 4.2 define the action of the groups $\mathfrak{S}_r$ and $\mathfrak{S}_s$ on the K-module $T_{(r,s)}$ composed of the tensor fields of the type (r,s). This is the action by linear operators, i. e.

$$\begin{aligned}&\sigma\circ\tau(\mathbf{A}+\mathbf{B})=\sigma\circ\tau(\mathbf{A})+\sigma\circ\tau(\mathbf{B}),\\ &\sigma\circ\tau(\xi\cdot A)=\xi\cdot(\sigma\circ\tau(\mathbf{A}))\end{aligned}\tag{4.3}$$

for any two transpositions $\sigma\in\mathfrak{S}_r$ and $\tau\in\mathfrak{S}_s$. When written in coordinate form, the relationship $\mathbf{B}=\sigma\circ\tau(\mathbf{A})$ looks like

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=A^{i_{\sigma(1)}\ldots\,i_{\sigma(r)}}_{j_{\tau(1)}\ldots\,j_{\tau(s)}},\tag{4.4}$$

where the umbers $\sigma(1),\ldots,\sigma(r)$ and $\tau(1),\ldots,\tau(s)$ are obtained by applying σ and τ to the numbers $1,\ldots,r$ and $1,\ldots,s$.

Definition 4.1. A tensorial field $\mathbf{A}$ of the type (r, s) is said to be *symmetric* in m-th and n-th upper (or lower) indices if $\sigma(\mathbf{A}) = \mathbf{A}$, where σ is the permutation of the indices given by the formula (4.1) (or the formula (4.2)).

Definition 4.2. A tensorial field $\mathbf{A}$ of the type (r, s) is said to be *skew-symmetric* in m-th and n-th upper (or lower) indices if $\sigma(\mathbf{A}) = -\mathbf{A}$, where σ is the permutation of the indices given by the formula (4.1) (or the formula (4.2)).

The concepts of symmetry and skew-symmetry can be extended to the case of arbitrary (not necessarily pairwise) transpositions. Let $\varepsilon = \sigma \circ \tau$ be some transposition of upper and lower indices from (4.4). It is natural to treat it as an element of direct product of two symmetric groups: $\varepsilon \in \mathfrak{S}_r \times \mathfrak{S}_s$ (see [3]).

Definition 4.3. A tensorial field $\mathbf{A}$ of the type (r, s) is *symmetric* or *skew-symmetric* with respect to the transposition $\varepsilon \in \mathfrak{S}_r \times \mathfrak{S}_s$, if one of the following relationships is fulfilled: $\varepsilon(\mathbf{A}) = \mathbf{A}$ or $\varepsilon(\mathbf{A}) = (-1)^{\varepsilon} \cdot \mathbf{A}$.

If the field $\mathbf{A}$ is symmetric with respect to the transpositions ε_1 and ε_2, then it is symmetric with respect to the composite transposition $\varepsilon_1 \circ \varepsilon_2$ and with respect to the inverse transpositions ε_1^{-1} and ε_2^{-1}. Therefore the symmetry always takes place for some subgroup $\mathfrak{G} \in \mathfrak{S}_r \times \mathfrak{S}_s$. The same is true for the skew-symmetry.

Let $\mathfrak{G} \subset \mathfrak{S}_r \times \mathfrak{S}_s$ be a subgroup in the direct product of symmetric groups and let $\mathbf{A}$ be a tensor field from $T_{(r,s)}$. The passage from $\mathbf{A}$ to the field

$$\mathbf{B} = \frac{1}{|\mathfrak{G}|} \sum_{\varepsilon \in \mathfrak{G}} \varepsilon(\mathbf{A}) \tag{4.5}$$

is called the *symmetrization* of the tensor field $\mathbf{A}$ by the subgroup $\mathfrak{G} \subset \mathfrak{S}_r \times \mathfrak{S}_s$. Similarly, the passage from $\mathbf{A}$ to the field

$$\mathbf{B} = \frac{1}{|\mathfrak{G}|} \sum_{\varepsilon \in \mathfrak{G}} (-1)^{\varepsilon} \cdot \varepsilon(\mathbf{A}) \tag{4.6}$$

is called the *alternation* of the tensor field $\mathbf{A}$ by the subgroup $\mathfrak{G} \subset \mathfrak{S}_r \times \mathfrak{S}_s$.

The operations of symmetrization and alternation are linear operations, this fact follows from (4.3). As a result of symmetrization (4.5) one gets a field $\mathbf{B}$ symmetric with respect to $\mathfrak{G}$. As a result of alternation (4.6) one gets a field skew-symmetric with respect to $\mathfrak{G}$. If $\mathfrak{G} = \mathfrak{S}_r \times \mathfrak{S}_s$ then the operation (4.5) is called the *complete symmetrization*, while the (4.6) is called the *complete alternation*.

§ 5. Differentiation of tensor fields.

The smoothness class of a tensor field $\mathbf{A}$ in the space $\mathbb{E}$ is determined by the smoothness of its components.

Definition 5.1. A tensor field $\mathbf{A}$ is called an *m-times continuously differentiable field* or a *field of the class C^m* if all its components in some Cartesian system are m-times continuously differentiable functions.

Tensor fields of the class C^1 are often called *differentiable tensor fields*, while fields of the class C^∞ are called *smooth tensor fields*. Due to the formula (1.7) the choice of a Cartesian coordinate system does not affect the smoothness class of a field $\mathbf{A}$ in the definition 5.1. The components of a field of the class C^m are the functions of the class C^m in any Cartesian coordinate system. This fact proves that the definition 5.1 is consistent.

Let's consider a differentiable tensor field of the type (r, s) and let's consider all of the partial derivatives of its components:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s\,j_{s+1}} = \frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^{j_{s+1}}}. \tag{5.1}$$

The number of such partial derivatives (5.1) is the same in all Cartesian coordinate systems. This number coincides with the number of components of a tensor field of the type $(r, s+1)$. This coincidence is not accidental.

THEOREM 5.1. *The partial derivatives of the components of a differentiable tensor field* $\mathbf{A}$ *of the type* (r, s) *calculated in an arbitrary Cartesian coordinate system according to the formula* (5.1) *are the components of another tensor filed* $\mathbf{B}$ *of the type* $(r, s+1)$.

PROOF. The proof consists in checking up the transformation rule (1.7) for the quantities $B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s\,j_{s+1}}$ in (5.1). Let $O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $O', \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ be two Cartesian coordinate systems. By tradition we denote by S and T the direct and inverse transition matrices. Let's write the first relationship (1.7) for the field $\mathbf{A}$ and let's differentiate both sides of it with respect to the variable $x^{j_{s+1}}$:

$$\frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(\mathbf{x})}{\partial x^{j_{s+1}}} = \sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\,\frac{\partial \tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}(\tilde{\mathbf{x}})}{\partial x^{j_{s+1}}}$$

In order to calculate the derivative in the right hand side we apply the chain rule that determines the derivatives of a composite function:

$$\frac{\partial \tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}(\tilde{\mathbf{x}})}{\partial x^{j_{s+1}}} = \sum_{q_{s+1}=1}^{3} \frac{\partial \tilde{x}^{q_{s+1}}}{\partial x^{j_{s+1}}}\,\frac{\partial \tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}(\tilde{\mathbf{x}})}{\partial \tilde{x}^{q_{s+1}}}. \tag{5.2}$$

The variables $\mathbf{x} = (x^1, x^2, x^3)$ and $\tilde{\mathbf{x}} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ are related as follows:

$$x^i = \sum_{j=1}^{3} S^i_j\,\tilde{x}^j + a^i, \qquad\qquad \tilde{x}^i = \sum_{j=1}^{3} T^i_j\,x^j + \tilde{a}^i.$$

One of these two relationships is included into (1.7), the second being the inversion of the first one. The components of the transition matrices S and T in these formulas are constants, therefore, we have

$$\frac{\partial \tilde{x}^{q_{s+1}}}{\partial x^{j_{s+1}}} = T^{q_{s+1}}_{j_{s+1}}. \tag{5.3}$$

Let's substitute (5.3) into (5.2), then substitute the result into the above expression for the derivatives $\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}/\partial x^{j_{s+1}}$. This yields the equality

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s\,j_{s+1}} = \sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_{s+1}}} S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_{s+1}}_{j_{s+1}}\,\tilde{B}^{p_1\ldots\,p_r}_{q_1\ldots\,j_{s+1}}$$

which coincides exactly with the transformation rule (1.7) applied to the quantities (5.1). The theorem is proved. □

The passage from $\mathbf{A}$ to $\mathbf{B}$ in (5.1) adds one covariant index j_{s+1}. This is the reason why the tensor field $\mathbf{B}$ is called the *covariant differential* of the field $\mathbf{A}$. The covariant differential is denoted as $\mathbf{B} = \nabla\mathbf{A}$. The upside-down triangle ∇ is a special symbol, it is called *nabla*. In writing the components of $\mathbf{B}$ the additional covariant index is written beside the nabla sign:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s\,k} = \nabla_k A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}. \tag{5.4}$$

Due to (5.1) the sign ∇_k in the formula (5.4) replaces the differentiation operator: $\nabla_k = \partial/\partial x^k$. However, for ∇_k the special name is reserved, it is called the operator of *covariant differentiation* or the *covariant derivative*. Below (in Chapter III) we shall see that the concept of covariant derivative can be extended so that it will not coincide with the partial derivative any more.

Let $\mathbf{A}$ be a differentiable tensor field of the type (r, s) and let $\mathbf{X}$ be some arbitrary vector field. Let's consider the tensor product $\nabla\mathbf{A} \otimes \mathbf{X}$. This is the tensor field of the type $(r + 1, s + 1)$. The covariant differentiation adds one covariant index, while the tensor multiplication add one contravariant index. We denote by $\nabla_{\mathbf{X}}\mathbf{A} = C(\nabla\mathbf{A} \otimes \mathbf{X})$ the contraction of the field $\nabla\mathbf{A} \otimes \mathbf{X}$ with respect to these two additional indices. The field $\mathbf{B} = \nabla_{\mathbf{X}}\mathbf{A}$ has the same type (r, s) as the original field $\mathbf{A}$. Upon choosing some Cartesian coordinate system we can write the relationship $\mathbf{B} = \nabla_{\mathbf{X}}\mathbf{A}$ in coordinate form:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum_{q=1}^{3} X^q\,\nabla_q A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}. \tag{5.5}$$

The tensor field $\mathbf{B} = \nabla_{\mathbf{X}}\mathbf{A}$ with components (5.5) is called the *covariant derivative of the field* $\mathbf{A}$ *along the vector field* $\mathbf{X}$.

THEOREM 5.2. *The operation of covariant differentiation of tensor fields possesses the following properties*

(1) $\nabla_{\mathbf{X}}(\mathbf{A} + \mathbf{B}) = \nabla_{\mathbf{X}}\mathbf{A} + \nabla_{\mathbf{X}}\mathbf{B}$;
(2) $\nabla_{\mathbf{X}+\mathbf{Y}}\mathbf{A} = \nabla_{\mathbf{X}}\mathbf{A} + \nabla_{\mathbf{Y}}\mathbf{A}$;
(3) $\nabla_{\xi\cdot\mathbf{X}}\mathbf{A} = \xi \cdot \nabla_{\mathbf{X}}\mathbf{A}$;
(4) $\nabla_{\mathbf{X}}(\mathbf{A} \otimes \mathbf{B}) = \nabla_{\mathbf{X}}\mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \nabla_{\mathbf{X}}\mathbf{B}$;
(5) $\nabla_{\mathbf{X}}C(\mathbf{A}) = C(\nabla_{\mathbf{X}}\mathbf{A})$;

where $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary differentiable tensor fields, while* $\mathbf{X}$ *and* $\mathbf{Y}$ *are arbitrary vector fields and* ξ *is an arbitrary scalar field.*

PROOF. It is convenient to carry out the proof of the theorem in some Cartesian coordinate system. Let $\mathbf{C} = \mathbf{A} + \mathbf{B}$. The property (1) follows from the relationship

$$\sum_{q=1}^{3} X^q\,\frac{\partial C^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q} = \sum_{q=1}^{3} X^q\,\frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q} + \sum_{q=1}^{3} X^q\,\frac{\partial B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q}.$$

Denote $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ and then we derive the property (2) from the relationship

$$\sum_{q=1}^{3} Z^q\,\nabla_q A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum_{q=1}^{3} X^q\,\nabla_q A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} + \sum_{q=1}^{3} Y^q\,\nabla_q A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}.$$

In order to prove the property (3) we set $\mathbf{Z} = \xi \cdot \mathbf{X}$. Then

$$\sum_{q=1}^{3} Z^q \, \nabla_q A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \xi \sum_{q=1}^{3} X^q \, \nabla_q A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}.$$

This relationship is equivalent to the property (3) in the statement of the theorem.

In order to prove the fourth property in the theorem one should carry out the following calculations with the components of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$:

$$\sum_{q=1}^{3} X^q \, \partial/\partial x^q \left(A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} \, B^{i_{r+1} \ldots\, i_{r+m}}_{j_{s+1} \ldots\, j_{s+n}} \right) = \left(\sum_{q=1}^{3} X^q \, \frac{\partial A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial x^q} \right) \times$$

$$\times \, B^{i_{r+1} \ldots\, i_{r+m}}_{j_{s+1} \ldots\, j_{s+n}} + A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} \left(\sum_{q=1}^{3} X^q \, \frac{\partial B^{i_{r+1} \ldots\, i_{r+m}}_{j_{s+1} \ldots\, j_{s+n}}}{\partial x^q} \right).$$

And finally, the following series of calculations

$$\sum_{q=1}^{3} X^q \, \frac{\partial}{\partial x^q} \left(\sum_{k=1}^{3} A^{i_1 \ldots\, i_{m-1}\, k\, i_m \ldots\, i_{r-1}}_{j_1 \ldots\, j_{n-1}\, k\, j_n \ldots\, j_{s-1}} \right) =$$

$$= \sum_{k=1}^{3} \sum_{q=1}^{3} X^q \, \frac{\partial A^{i_1 \ldots\, i_{m-1}\, k\, i_m \ldots\, i_{r-1}}_{j_1 \ldots\, j_{n-1}\, k\, j_n \ldots\, j_{s-1}}}{\partial x^q}$$

proves the fifth property. This completes the proof of the theorem in whole. □

§ 6. The metric tensor and the volume pseudotensor.

Let $O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be some Cartesian coordinate system in the space $\mathbb{E}$. The space $\mathbb{E}$ is equipped with the scalar product. Therefore, the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of any Cartesian coordinate system has its Gram matrix

$$g_{ij} = (\mathbf{e}_i \,|\, \mathbf{e}_j). \tag{6.1}$$

The gram matrix $\mathbf{g}$ is positive and non-degenerate:

$$\det \mathbf{g} > 0. \tag{6.2}$$

The inequality (6.2) follows from the Silvester criterion (see [1]). Under a change of a coordinate system the quantities (6.1) are transformed as the components of a tensor of the type $(0, 2)$. Therefore, we can define the tensor field $\mathbf{g}$ whose components in any Cartesian coordinate system are the constant functions coinciding with the components of the Gram matrix:

$$g_{ij}(\mathbf{x}) = g_{ij} = \text{const}\,.$$

The tensor field $\mathbf{g}$ with such components is called the *metric tensor*. The metric tensor is a special tensor field. One should not define it. Its existence is providentially built into the geometry of the space $\mathbb{E}$.

Since the Gram matrix $\mathbf{g}$ is non-degenerate, one can determine the inverse matrix $\hat{\mathbf{g}} = \mathbf{g}^{-1}$. The components of such matrix are denoted by g^{ij}, the indices i and j are written in the upper position. Then

$$\sum_{j=1}^{3} g^{ij} \, g_{jk} = \delta^i_j. \tag{6.3}$$

Theorem 6.1. *The components of the inverse Gram matrix* $\hat{\mathbf{g}}$ *are transformed as the components of a tensor field of the type* $(2, 0)$ *under a change of coordinates.*

Proof. Let's write the transformation rule (1.7) for the components of the metric tensor $\mathbf{g}$:

$$g_{ij} = \sum_{p=1}^{3} \sum_{q=1}^{3} T^p_i \, T^q_j \, \tilde{g}_{pq}.$$

In matrix form this relationship is written as

$$\mathbf{g} = T^{\mathrm{tr}} \, \tilde{\mathbf{g}} \, T. \tag{6.4}$$

Since $\mathbf{g}$, $\tilde{\mathbf{g}}$, and T are non-degenerate, we can pass to the inverse matrices:

$$\mathbf{g}^{-1} = (T^{\mathrm{tr}} \, \tilde{\mathbf{g}} \, T)^{-1} = S \, \tilde{\mathbf{g}}^{-1} \, S^{\mathrm{tr}}. \tag{6.5}$$

Now we can write (6.5) back in coordinate form. This yields

$$g^{ij} = \sum_{p=1}^{3} \sum_{q=1}^{3} S^i_p \, S^j_q \, \tilde{g}^{pq}. \tag{6.6}$$

The relationship (6.6) is exactly the transformation rule (1.7) written for the components of a tensor field of the type $(2, 0)$. Thus, the theorem is proved. □

The tensor field $\hat{\mathbf{g}} = \mathbf{g}^{-1}$ with the components g^{ij} is called the *inverse metric tensor* or the *dual metric tensor*. The existence of the inverse metric tensor also follows from the nature of the space $\mathbf{E}$ which has the pre-built scalar product.

Both tensor fields $\mathbf{g}$ and $\hat{\mathbf{g}}$ are symmetric. The symmetruy of g_{ij} with respect to the indices i and j follows from (6.1) and from the properties of a scalar product. The matrix inverse to the symmetric one is a symmetric matrix too. Therefore, the components of the inverse metric tensor g^{ij} are also symmetric with respect to the indices i and j.

The components of the tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$ in any Cartesian coordinate system are constants. Therefore, we have

$$\nabla \mathbf{g} = 0, \qquad\qquad \nabla \hat{\mathbf{g}} = 0. \tag{6.7}$$

These relationships follow from the formula (5.1) for the components of the covariant differential in Cartesian coordinates.

In the course of analytical geometry (see, for instance, [4]) the indexed object ε_{ijk} is usually considered, which is called the Levi-Civita symbol. Its nonzero components are determined by the parity of the transposition of indices:

$$\varepsilon_{ijk} = \varepsilon^{ijk} = \begin{cases} 0 & \text{if } i = j, \ \ i = k, \ \text{ or } \ j = k, \\ 1 & \text{if } (ijk) \text{ is even, i.e. } \operatorname{sign}(ijk) = 1, \\ -1 & \text{if } (ijk) \text{ is odd, i.e. } \operatorname{sign}(ijk) = -1. \end{cases} \tag{6.8}$$

Recall that the Levi-Civita symbol (6.8) is used for calculating the vectorial product[1] and the mixed product[2] through the coordinates of vectors in a rectangular Cartesian coordinate system with a right orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\begin{aligned}[\mathbf{X},\,\mathbf{Y}] &= \sum_{i=1}^{3}\mathbf{e}_i\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,X^j\,Y^k\right),\\ (\mathbf{X},\,\mathbf{Y},\,\mathbf{Z}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,X^i\,Y^j\,Z^k.\end{aligned}\tag{6.9}$$

The usage of upper or lower indices in writing the components of the Levi-Civita symbol in (6.8) and (6.9) makes no difference since they do not define a tensor. However, there is a tensorial object associated with the Levi-Civita symbol. In order to construct such an object we apply the relationship which is usually proved in analytical geometry:

$$\sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{l=1}^{3}\varepsilon_{pql}\,M_{ip}\,M_{jq}\,M_{kl} = \det\mathbf{M}\cdot\varepsilon_{ijk}\tag{6.10}$$

(see proof in [4]). Here $\mathbf{M}$ is some square 3×3 matrix. The matrix $\mathbf{M}$ can be the matrix of the components for some tensorial field of the type $(2,0)$, $(1,1)$, or $(0,2)$. However, it can be a matrix without any tensorial interpretation as well. The relationship (6.10) is valid for any square 3×3 matrix.

Using the Levi-Civita symbol and the matrix of the metric tensor $\mathbf{g}$ in some Cartesian coordinate system, we construct the following quantities:

$$\omega_{ijk} = \sqrt{\det\mathbf{g}}\;\varepsilon_{ijk}.\tag{6.11}$$

Then we study how the quantities ω_{ijk} and $\tilde\omega_{pql}$ constructed in two different Cartesian coordinate systems O, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and O', $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$ are related to each other. From the identity (6.10) we derive

$$\sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{l=1}^{3}T^p_i\,T^q_j\,T^l_k\,\tilde\omega_{pql} = \sqrt{\det\tilde{\mathbf{g}}}\;\det T\;\varepsilon_{ijk}.\tag{6.12}$$

In order to transform further the sum (6.12) we use the relationship (6.4), as an immediate consequence of it we obtain the formula $\det\mathbf{g} = (\det T)^2\,\det\tilde{\mathbf{g}}$. Applying this formula to (6.12), we derive

$$\sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{l=1}^{3}T^p_i\,T^q_j\,T^l_k\,\tilde\omega_{pql} = \operatorname{sign}(\det T)\;\sqrt{\det\mathbf{g}}\;\varepsilon_{ijk}.\tag{6.13}$$

Note that the right hand side of the relationship (6.13) differs from ω_{ijk} in (6.11) only by the sign of the determinant: $\operatorname{sign}(\det T) = \operatorname{sign}(\det S) = \pm 1$. Therefore, we can write the relationship (6.13) as

$$\omega_{ijk} = \operatorname{sign}(\det S)\;\sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{l=1}^{3}T^p_i\,T^q_j\,T^l_k\,\tilde\omega_{pql}.\tag{6.14}$$

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[1] It is also called the *cross product* of vectors.

[2] The mixed product is defined as $(\mathbf{X},\,\mathbf{Y},\,\mathbf{Z}) = (\mathbf{X}\,|\,[\mathbf{Y},\,\mathbf{Z}])$.

Though the difference is only in sign, the relationship (6.14) differs from the transformation rule (1.6) for the components of a tensor of the type $(0,3)$. The formula (6.14) gives the cause for modifying the transformation rule (1.6):

$$F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}} (-1)^S\, S^{i_1}_{p_1} \ldots S^{i_r}_{p_r}\, T^{q_1}_{j_1} \ldots T^{q_s}_{j_s}\, \tilde{F}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}. \tag{6.15}$$

Here $(-1)^S = \operatorname{sign}(\det S) = \pm 1$. The corresponding modification for the concept of a tensor is given by the following definition.

Definition 6.1. A *pseudotensor* $\mathbf{F}$ of the type (r,s) is a geometric object whose components in an arbitrary basis are enumerated by $(r+s)$ indices and obey the transformation rule (6.15) under a change of basis.

Once some pseudotensor of the type (r,s) is given at each point of the space $\mathbb{E}$, we have a pseudotensorial field of the type (r,s). Due to the above definition 6.1 and due to (6.14) the quantities ω_{ijk} from (6.11) define a pseudotensorial field $\boldsymbol{\omega}$ of the type $(0,3)$. This field is called the *volume pseudotensor*. Like metric tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$, the volume pseudotensor is a special field pre-built into the space $\mathbb{E}$. Its existence is due to the existence of the pre-built scalar product in $\mathbb{E}$.

§ 7. The properties of pseudotensors.

Pseudotensors and pseudotensorial fields are closely relative objects for tensors and tensorial fields. In this section we repeat most of the results of previous sections as applied to pseudotensors. The proofs of these results are practically the same as in purely tensorial case. Therefore, below we do not give the proofs.

Let $\mathbf{A}$ and $\mathbf{B}$ be two pseudotensorial fields of the type (r,s). Then the formula (3.1) determines a third field $\mathbf{C} = \mathbf{A} + \mathbf{B}$ which appears to be a pseudotensorial field of the type (r,s). It is important to note that (3.1) is not a correct procedure if one tries to add a tensorial field $\mathbf{A}$ and a pseudotensorial field $\mathbf{B}$. The sum $\mathbf{A} + \mathbf{B}$ of such fields can be understood only as a formal sum like in (3.5).

The formula (2.2) for the tensor product appears to be more universal. It defines the product of a field $\mathbf{A}$ of the type (r,s) and a field $\mathbf{B}$ of the type (m,n). Therein each of the fields can be either a tensorial or a pseudotensorial field. The tensor product possesses the following properties:

(1) the tensor product of two tensor fields is a tensor field;
(2) the tensor product of two pseudotensorial fields is a tensor field;
(3) the tensor product of a tensorial field and a pseudotensorial field is a pseudotensorial field.

Let's denote by $P_{(r,s)}$ the set of pseudotensorial fields of the type (r,s). Due to the properties (1)-(3) and due to the distributivity relationships (3.3), which remain valid for pseudotensorial fields too, the set $P_{(r,s)}$ is a module over the ring of scaral fields $K = T_{(0,0)}$. As for the properties (1)-(3), they can be expressed in form of the relationships

$$\begin{aligned}
&T_{(r,s)} \times T_{(m,n)} \to T_{(r+m,s+n)}, &\qquad &T_{(r,s)} \times P_{(m,n)} \to P_{(r+m,s+n)},\\
&P_{(r,s)} \times P_{(m,n)} \to T_{(r+m,s+n)}, &\qquad &P_{(r,s)} \times T_{(m,n)} \to P_{(r+m,s+n)}
\end{aligned} \tag{7.1}$$

that extend the relationship (3.2) from the section 3.

The formula (2.5) defines the operation of contraction for a field $\mathbf{F}$ of the type (r, s), where $r \geqslant 1$ and $s \geqslant 1$. The operation of contraction (2.5) is applicable to tensorial and pseudotensorial fields. It has the following properties:

(1) the contraction of a tensorial field is a tensorial field;
(2) the contraction of a pseudotensorial field is a pseudotensorial field.

The operation of contraction extended to the case of pseudotensorial fields preserve its linearity given by the equalities (3.4).

The covariant differentiation of pseudotensorial fields in a Cartesian coordinate system is determined by the formula (5.1). The covariant differential $\nabla\mathbf{A}$ of a tensorial field is a tensorial field; the covariant differential of a pseudotensorial field is a pseudotensorial field. It is convenient to express the properties of the covariant differential through the properties of the covariant derivative $\nabla_{\mathbf{X}}$ in the direction of a field $\mathbf{X}$. Now $\mathbf{X}$ is either a vectorial or a pseudovectorial field. All propositions of the theorem 5.2 for $\nabla_{\mathbf{X}}$ remain valid.

§ 8. A note on the orientation.

Pseudoscalar fields form a particular case of pseudotensorial fields. Scalar fields can be interpreted as functions whose argument is a point of the space $\mathbb{E}$. In this interpretation they do not depend on the choice of a coordinate system. Pseudoscalar fields even in such interpretation preserve some dependence on a coordinate system, though this dependence is rather weak. Let ξ be a pseudoscalar field. In a fixed Cartesian coordinate system the field ξ is represented by a scalar function $\xi(P)$, where $P \in \mathbb{E}$. The value of this function ξ at a point P does not change if we pass to another coordinate system of the same orientation, i. e. if the determinant of the transition matrix S is positive. When passing to a coordinate system of the opposite orientation the function ξ changes the sign: $\xi(P) \to -\xi(P)$. Let's consider a nonzero constant pseudoscalar field ξ. In some coordinate systems $\xi = c = \text{const}$, in others $\xi = -c = \text{const}$. Without loss of generality we can take $c = 1$. Then such a pseudoscalar field ξ can be used to distinguish the coordinate systems where $\xi = 1$ from those of the opposite orientation where $\xi = -1$.

Proposition. *Defining a unitary constant pseudoscalar field ξ is equivalent to choosing some preferable orientation in the space $\mathbb{E}$.*

From purely mathematical point of view the space $\mathbb{E}$, which is a three-dimensional Euclidean point space (see definition in [1]), has no preferable orientation. However, the real physical space $\mathbb{E}$ (where we all live) has such an orientation. Therein we can distinguish the left hand from the right hand. This difference in the nature is not formal and purely terminological: the left hemisphere of a human brain is somewhat different from the right hemisphere in its functionality, in many substances of the organic origin some isomers prevail over the mirror symmetric isomers. The number of left-handed people and the number of right-handed people in the mankind is not fifty-fifty as well. The asymmetry of the left and right is observed even in basic forms of the matter: it is reflected in modern theories of elementary particles. Thus, we can assume the space $\mathbb{E}$ to be canonically equipped with some pseudoscalar field ξ_E whose values are given by the formula

$$\xi_E = \begin{cases} 1 & \text{in right-oriented coordinate systems,} \\ -1 & \text{in left-oriented coordinate systems.} \end{cases}$$

Multiplying by ξ_E, we transform a tensorial field $\mathbf{A}$ into the pseudotensorial field $\xi_E \otimes \mathbf{A} = \xi_E \cdot \mathbf{A}$. Multiplying by ξ_E once again, we transform $\xi_E \cdot \mathbf{A}$ back to $\mathbf{A}$. Therefore, in the space $\mathbb{E}$ equipped with the preferable orientation in form of the field ξ_E one can not to consider pseudotensors at all, considering only tensor fields. The components of the volume tensor $\boldsymbol{\omega}$ in this space should be defined as

$$\omega_{ijk} = \xi_E \sqrt{\det \mathbf{g}}\ \varepsilon_{ijk}. \tag{8.1}$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be two vectorial fields. Then we can define the vectorial field $\mathbf{Z}$ with the following components:

$$Z^q = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} g^{ki}\,\omega_{ijk}\ X^j\,Y^k. \tag{8.2}$$

From (8.2), it is easy to see that $\mathbf{Z}$ is derived as the contraction of the field $\hat{\mathbf{g}} \otimes \boldsymbol{\omega} \otimes \mathbf{a} \otimes \mathbf{b}$. In a rectangular Cartesian coordinate system with right-oriented orthonormal basis the formula (8.2) takes the form of the well-known formula for the components of the vector product $\mathbf{Z} = [\mathbf{X},\,\mathbf{Y}]$ (see [4] and the formula (6.9) above). In a space without preferable orientation, where ω_{ijk} is given by the formula (6.11), the vector product of two vectors is a pseudovector.

Now let's consider three vectorial fields $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ and let's construct the scalar field u by means of the following formula:

$$u = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \omega_{ijk}\ X^i\,Y^j\,Z^k. \tag{8.3}$$

When (8.3) is written in a rectangular Cartesian coordinate system with right-oriented orthonormal basis, one easily sees that the field (8.3) coincides with the mixed product $(\mathbf{X},\,\mathbf{Y}\,\mathbf{Z})$ of three vectors (see [4] and the formula (6.9) above). In a space without preferable orientation the mixed product of three vector fields determined by the volume pseudotensor (6.11) appears to be a pseudoscalar field.

§ 9. Raising and lowering indices.

Let $\mathbf{A}$ be a tensor field or a pseudotensor field of the type (r,s) in the space $\mathbb{E}$ and let $r \geqslant 1$. Let's construct the tensor product $\mathbf{A} \otimes \mathbf{g}$ of the field A and the metric tensor $\mathbf{g}$, then define the field $\mathbf{B}$ of the type $(r-1, s+1)$ as follows:

$$B^{i_1\ldots\, i_{r-1}}_{j_1\ldots\, j_{s+1}} = \sum_{k=1}^{3} A^{i_1\ldots\, i_{m-1}\,k\,i_m\ldots\, i_{r-1}}_{j_1\ldots\, j_{n-1}\,j_{n+1}\ldots\, j_{s+1}}\ g_{kj_n}. \tag{9.1}$$

The passage from the field $\mathbf{A}$ to the field $\mathbf{B}$ according to the formula (9.1) is called the *index lowering procedure of the m-th upper index to the n-th lower position.*

Using the inverse metric tensor, one can invert the operation (9.1). Let $\mathbf{B}$ be a tensorial or a pseudotensorial field of the type (r,s) and let $s \geqslant 1$. Then we define the field $\mathbf{A} = C(\mathbf{B} \otimes \hat{\mathbf{g}})$ of the type $(r+1, s-1)$ according to the formula:

$$A^{i_1 \ldots\, i_{r+1}}_{j_1 \ldots\, j_{s-1}} = \sum_{q=1}^{3} B^{i_1 \ldots\, i_{m-1}\, i_{m+1} \ldots\, i_{r+1}}_{j_1 \ldots\, j_{n-1}\, q\, j_n \ldots\, j_{s-1}}\; g^{q i_m}. \tag{9.2}$$

The passage from the field $\mathbf{B}$ to the field $\mathbf{A}$ according to the formula (9.2) is called the *index raising procedure of the n-th lower index to the m-th upper position*.

The operations of lowering and raising indices are inverse to each other. Indeed, we can perform the following calculations:

$$\begin{aligned} C^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} &= \sum_{q=1}^{3} \sum_{k=1}^{3} A^{i_1 \ldots\, i_{m-1}\, k\, i_{m+1} \ldots\, i_r}_{j_1 \ldots\, j_{n-1}\, j_n \ldots\, j_s}\; g_{kq}\, g^{q i_m} = \\ &= \sum_{k=1}^{3} A^{i_1 \ldots\, i_{m-1}\, k\, i_{m+1} \ldots\, i_r}_{j_1 \ldots\, j_{n-1}\, j_n \ldots\, j_s}\; \delta^{i_m}_{k} = A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}. \end{aligned}$$

The above calculations show that applying the index lowering and the index raising procedures successively $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$, we get the field $\mathbf{C} = \mathbf{A}$. Applying the same procedures in the reverse order yields the same result. This follows from the calculations just below:

$$\begin{aligned} C^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} &= \sum_{k=1}^{3} \sum_{q=1}^{3} A^{i_1 \ldots\, i_{m-1}\, i_m \ldots\, i_r}_{j_1 \ldots\, j_{n-1}\, q\, j_{n+1} \ldots\, j_s}\; g^{qk}\, g_{k j_n} = \\ &= \sum_{q=1}^{3} A^{i_1 \ldots\, i_{m-1}\, i_m \ldots\, i_r}_{j_1 \ldots\, j_{n-1}\, q\, j_{n+1} \ldots\, j_s}\; \delta^{q}_{i_n} = A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}. \end{aligned}$$

The existence of the index raising and index lowering procedures follows from the very nature of the space $\mathbb{E}$ which is equipped with the scalar product and, hence, with the metric tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$. Therefore, any tensorial (or pseudotensorial) field of the type (r,s) in such a space can be understood as originated from some purely covariant field of the type $(0, r+s)$ as a result of raising a part of its indices. Therein a little bit different way of setting indices is used. Let's consider a field $\mathbf{A}$ of the type $(0,4)$ as an example. We denote by $A_{i_1 i_2 i_3 i_4}$ its components in some Cartesian coordinate system. By raising one of the four indices in $\mathbf{A}$ one can get the four fields of the type $(1,3)$. Their components are denoted as

$$A^{i_1}{}_{i_2 i_3 i_4}, \quad A_{i_1}{}^{i_2}{}_{i_3 i_4}, \quad A_{i_1 i_2}{}^{i_3}{}_{i_4}, \quad A_{i_1 i_2 i_3}{}^{i_4}. \tag{9.3}$$

Raising one of the indices in (9.3), we get an empty place underneath it in the list of lower indices, while the numbering of the indices at that place remains unbroken. In this way of writing indices, each index has «its fixed position» no matter what index it is — a lower or an upper index. Therefore, in the writing below we easily guess the way in which the components of tensors are derived:

$$A^{i}{}_{jkq},\ A^{ij}{}_{kq},\ A^{i}{}_{j}{}^{k}{}_{q},\ A_{i}{}^{jk}{}_{q}. \tag{9.4}$$

Despite to some advantages of the above form of index setting in (9.3) and (9.4), it is not commonly admitted. The matter is that it has a number of disadvantages either. For example, the writing of general formulas (1.6), (2.2), (2.5), and some others becomes huge and inconvenient for perception. In what follows we shall not change the previous way of index setting.

§ 10. Gradient, divergency, and rotor. Some identities of the vectorial analysis.

Let's consider a scalar field or, in other words, a function f. Then apply the operator of covariant differentiation ∇, as defined in § 5, to the field f. The covariant differential ∇f is a covectorial field (a field of the type $(0,1)$). Applying the index raising procedure (9.2) to the covector field ∇f, we get the vector field $\mathbf{F}$. Its components are given by the following formula:

$$F^i = \sum_{k=1}^{3} g^{ik}\,\frac{\partial f}{\partial x^k}. \tag{10.1}$$

Definition 10.1. The vector field $\mathbf{F}$ in the space $\mathbb{E}$ whose components are calculated by the formula (10.1) is called the *gradient* of a function f. The gradient is denoted as $\mathbf{F} = \operatorname{grad} f$.

Let $\mathbf{X}$ be a vectorial field in $\mathbb{E}$. Let's consider the scalar product of the vectorial fields $\mathbf{X}$ and $\operatorname{grad} f$. Due to the formula (10.1) such scalar product of two vectors is reduced to the scalar product of the vector $\mathbf{X}$ and the covector ∇f:

$$(\mathbf{X}\,|\,\operatorname{grad} f) = \sum_{k=1}^{3} X^k\,\frac{\partial f}{\partial x^k} = \langle \nabla f\,|\,\mathbf{X}\rangle. \tag{10.2}$$

The quantity (10.2) is a scalar quantity. It does not depend on the choice of a coordinate system where the components of $\mathbf{X}$ and ∇f are given. Another form of writing the formula (10.2) is due to the covariant differentiation along the vector field $\mathbf{X}$, it was introduced by the formula (5.5) above:

$$(\mathbf{X}\,|\,\operatorname{grad} f) = \nabla_{\mathbf{X}} f. \tag{10.3}$$

By analogy with the formula (10.3), the covariant differential $\nabla\mathbf{F}$ of an arbitrary tensorial field $\mathbf{F}$ is sometimes called the *covariant gradient* of the field $\mathbf{F}$.

Let $\mathbf{F}$ be a vector field. Then its covariant differential $\nabla\mathbf{F}$ is an operator field, i. e. a field of the type $(1,1)$. Let's denote by φ the contraction of the field $\nabla\mathbf{F}$:

$$\varphi = C(\nabla\mathbf{F}) = \sum_{k=1}^{3} \frac{\partial F^k}{\partial x^k}. \tag{10.4}$$

Definition 10.2. The scalar field φ in the space $\mathbb{E}$ determined by the formula (10.4) is called the *divergency* of a vector field $\mathbf{F}$. It is denoted $\varphi = \operatorname{div}\mathbf{F}$.

Apart from the scalar field $\operatorname{div}\mathbf{F}$, one can use $\nabla\mathbf{F}$ in order to build a vector field. Indeed, let's consider the quantities

$$\rho^m = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{mi}\,\omega_{ijk}\,g^{jq}\,\nabla_q F^k, \tag{10.5}$$

where ω_{ijk} are the components of the volume tensor given by the formula (8.1).

DEFINITION 10.3. The vector field $\boldsymbol{\rho}$ in the space $\mathbb{E}$ determined by the formula (10.5) is called the *rotor*[1] of a vector field $\mathbf{F}$. It is denoted $\boldsymbol{\rho} = \operatorname{rot}\mathbf{F}$.

Due to (10.5) the rotor or a vector field $\mathbf{F}$ is the contraction of the tensor field $\hat{\mathbf{g}} \otimes \boldsymbol{\omega} \otimes \hat{\mathbf{g}} \otimes \nabla\mathbf{F}$ with respect to four pairs of indices: $\operatorname{rot}\mathbf{F} = C(\hat{\mathbf{g}} \otimes \boldsymbol{\omega} \otimes \hat{\mathbf{g}} \otimes \nabla\mathbf{F})$.

Remark. If ω_{ijk} in (10.5) are understood as components of the volume pseudotensor (6.11), then the rotor of a vector field should be understood as a pseudovectorial field.

Suppose that O, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a rectangular Cartesian coordinate system in $\mathbb{E}$ with orthonormal right-oriented basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The Gram matrix of the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the unit matrix. Therefore, we have

$$g_{ij} = g^{ij} = \delta^i_j = \begin{cases} 1 & \text{for } \ i = j, \\ 0 & \text{for } \ i \neq j. \end{cases}$$

The pseudoscalar field ξ_E defining the orientation in $\mathbb{E}$ is equal to unity in a right-oriented coordinate system: $\xi_E \equiv 1$. Due to these circumstances the formulas (10.1) and (10.5) for the components of $\operatorname{grad} f$ and $\operatorname{rot}\mathbf{F}$ simplifies substantially:

$$(\operatorname{grad} f)^i = \frac{\partial f}{\partial x^i}, \tag{10.6}$$

$$(\operatorname{rot}\mathbf{F})^i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\,\frac{\partial F^k}{\partial x^j}. \tag{10.7}$$

The formula (10.4) for the divergency remains unchanged:

$$\operatorname{div}\mathbf{F} = \sum_{k=1}^{3} \frac{\partial F^k}{\partial x^k}. \tag{10.8}$$

The formula (10.7) for the rotor has an elegant representation in form of the determinant of a 3×3 matrix:

$$\operatorname{rot}\mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x^1} & \dfrac{\partial}{\partial x^2} & \dfrac{\partial}{\partial x^3} \\ F^1 & F^2 & F^3 \end{vmatrix}. \tag{10.9}$$

The formula (8.2) for the vector product in right-oriented rectangular Cartesian coordinate system takes the form of (6.9). It can also be represented in the form of the formal determinant of a 3×3 matrix:

$$[\mathbf{X},\,\mathbf{Y}] = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ X^1 & X^2 & X^3 \\ Y^1 & Y^2 & Y^3 \end{vmatrix}. \tag{10.10}$$

[1] The term «curl» is also used for the rotor.

Due to the similarity of (10.9) and (10.10) one can formally represent the operator of covariant differentiation ∇ as a vector with components $\partial/\partial x^1$, $\partial/\partial x^2$, $\partial/\partial x^3$. Then the divergency and rotor are represented as the scalar and vectorial products:

$$\operatorname{div}\mathbf{F} = (\nabla\,|\,\mathbf{F}), \qquad\qquad \operatorname{rot}\mathbf{F} = [\nabla,\,\mathbf{F}].$$

THEOREM 10.1. *For any scalar field φ of the smoothness class C^2 the equality* $\operatorname{rot}\operatorname{grad}\varphi = 0$ *is identically fulfilled.*

PROOF. Let's choose some right-oriented rectangular Cartesian coordinate system and then use the formulas (10.6) and(10.7). Let $\mathbf{F} = \operatorname{rot}\operatorname{grad}\varphi$. Then

$$F^i = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,\frac{\partial^2\varphi}{\partial x^j\,\partial x^k}. \tag{10.11}$$

Let's rename the summation indices in (10.11). The index j is replaced by the index k and vice versa. Such a swap of indices does not change the value of the sum in (10.11). Therefore, we have

$$F^i = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ikj}\,\frac{\partial^2\varphi}{\partial x^k\,\partial x^j} = -\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,\frac{\partial^2\varphi}{\partial x^j\,\partial x^k} = -F^i.$$

Here we used the skew-symmetry of the Levi-Civuta symbol with respect to the pair of indices j and k and the symmetry of the second order partial derivatives of the function φ with respect to the same pair of indices:

$$\frac{\partial^2\varphi}{\partial x^j\,\partial x^k} = \frac{\partial^2\varphi}{\partial x^k\,\partial x^j}. \tag{10.12}$$

For C^2 class functions the value of second order partial derivatives (10.12) do not depend on the order of differentiation. The equality $F^i = -F^i$ now immediately yields $F^i = 0$. The theorem is proved. $\square$

THEOREM 10.2. *For any vector field $\mathbf{F}$ of the smoothness class C^2 the equality* $\operatorname{div}\operatorname{rot}\mathbf{F} = 0$ *is identically fulfilled.*

PROOF. Here, as in the case of the theorem 10.1, we choose a right-oriented rectangular Cartesian coordinate system, then we use the formulas (10.7) and (10.8). For the scalar field $\varphi = \operatorname{div}\operatorname{rot}\mathbf{F}$ from these formulas we derive

$$\varphi = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,\frac{\partial^2 F^k}{\partial x^j\,\partial x^i}. \tag{10.13}$$

Using the relationship analogous to (10.12) for the partial derivatives

$$\frac{\partial^2 F^k}{\partial x^j\,\partial x^i} = \frac{\partial^2 F^k}{\partial x^i\,\partial x^j}$$

and using the skew-symmetry of ε_{ijk} with respect to indices i and j, from the formula (10.13) we easily derive $\varphi = -\varphi$. Hence, $\varphi = 0$. $\square$

Let φ be a scalar field of the smoothness class C^2. The quantity $\operatorname{div}\operatorname{grad}\varphi$ in general case is nonzero. It is denoted $\triangle\varphi = \operatorname{div}\operatorname{grad}\varphi$. The sign $\triangle$ denotes the differential operator of the second order that transforms a scalar field φ to another scalar field $\operatorname{div}\operatorname{grad}\varphi$. It is called the *Laplace operator* or the *laplacian*. In a rectangular Cartesian coordinate system it is given by the formula

$$\triangle = \left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2 + \left(\frac{\partial}{\partial x^3}\right)^2. \tag{10.14}$$

Using the formulas (10.6) and (10.8) one can calculate the Laplace operator in a skew-angular Cartesian coordinate system:

$$\triangle = \sum_{i=1}^{3}\sum_{j=1}^{3} g^{ij}\,\frac{\partial^2}{\partial x^i\,\partial x^j}. \tag{10.15}$$

Using the signs of covariant derivatives $\nabla_i = \partial/\partial x^i$ we can write the Laplace operator (10.15) as follows:

$$\triangle = \sum_{i=1}^{3}\sum_{j=1}^{3} g^{ij}\,\nabla_i\nabla_j. \tag{10.16}$$

The equality (10.16) differs from (10.15) not only in special notations for the derivatives. The Laplace operator defined as $\triangle\varphi = \operatorname{div}\operatorname{grad}\varphi$ can be applied only to a scalar field φ. The formula (10.16) extends it, and now we can apply the Laplace operator to any twice continuously differentiable tensor field $\mathbf{F}$ of any type (r, s). Due to this formula $\triangle\mathbf{F}$ is the result of contracting the tensor product $\hat{\mathbf{g}} \otimes \nabla\nabla\mathbf{F}$ with respect to two pairs of indices: $\triangle\mathbf{F} = C(\hat{\mathbf{g}} \otimes \nabla\nabla\mathbf{F})$. The resulting field $\triangle\mathbf{F}$ has the same type (r, s) as the original field $\mathbf{F}$. The laplace operator in the form of (10.16) is sometimes called the *Laplace-Beltrami* operator.

§ 11. Potential and vorticular vector fields.

Definition 11.1. A differentiable vector field $\mathbf{F}$ in the space $\mathbb{E}$ is called a *potential* field if $\operatorname{rot}\mathbf{F} = 0$.

Definition 11.2. A differentiable vector field $\mathbf{F}$ in the space $\mathbb{E}$ is called a *vorticular* field if $\operatorname{div}\mathbf{F} = 0$.

The theorem 10.1 yields some examples of potential vector fields, while the theorem 10.2 yields the examples of vorticular fields. Indeed, any field of the form $\operatorname{grad}\varphi$ is a potential field, and any field of the form $\operatorname{rot}\mathbf{F}$ is a vorticular one. As it appears, the theorems 10.1 and 10.2 can be strengthened.

Theorem 11.1. *Any potential vector field $\mathbf{F}$ in the space $\mathbb{E}$ is a gradient of some scalar field φ, i.e. $\mathbf{F} = \operatorname{grad}\varphi$.*

Proof. Let's choose a rectangular Cartesian coordinate system $O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with orthonormal right-oriented basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. In this coordinate system the potentiality condition $\operatorname{rot} \mathbf{F} = 0$ for the vector field $\mathbf{F}$ is equivalent tho the following three relationships for its components:

$$\frac{\partial F^1(x^1, x^2, x^3)}{\partial x^2} = \frac{\partial F^2(x^1, x^2, x^3)}{\partial x^1}, \tag{11.1}$$

$$\frac{\partial F^2(x^1, x^2, x^3)}{\partial x^3} = \frac{\partial F^3(x^1, x^2, x^3)}{\partial x^2}, \tag{11.2}$$

$$\frac{\partial F^3(x^1, x^2, x^3)}{\partial x^1} = \frac{\partial F^1(x^1, x^2, x^3)}{\partial x^3}. \tag{11.3}$$

The relationships (11.1), (11.2), and (11.3) are easily derived from (10.7) or from (10.9). Let's define the function $\varphi(x^1, x^2, x^3)$ as the sum of three integrals:

$$\begin{aligned} \varphi(x^1, x^2, x^3) = c + \int\limits_0^{x_1} F^1(x^1, 0, 0)\, dx^1 + \\ + \int\limits_0^{x_2} F^2(x^1, x^2, 0)\, dx^2 + \int\limits_0^{x_3} F^3(x^1, x^2, x^3)\, dx^3. \end{aligned} \tag{11.4}$$

Here c is an arbitrary constant. Now we only have to check up that the function (11.4) is that very scalar field for which $\mathbf{F} = \operatorname{grad} \varphi$.

Let's differentiate the function φ with respect to the variable x^3. The constant c and the first two integrals in (11.4) do not depend on x^3. Therefore, we have

$$\frac{\partial \varphi}{\partial x^3} = \frac{\partial}{\partial x^3} \int\limits_0^{x_3} F^3(x^1, x^2, x^3)\, dx^3 = F^3(x^1, x^2, x^3). \tag{11.5}$$

In deriving the relationship (11.5) we used the rule of differentiation of an integral with variable upper limit (see [2]).

Now let's differentiate the function φ with respect to x^2. The constant c and the first integral in (11.4) does not depend on x^2. Differentiating the rest two integrals, we get the following expression:

$$\frac{\partial \varphi}{\partial x^2} = F^2(x^1, x^2, 0) + \frac{\partial}{\partial x^2} \int\limits_0^{x_3} F^3(x^1, x^2, x^3)\, dx^3.$$

The operations of differentiation with respect to x^2 and integration with respect to x^3 in the above formula are commutative (see [2]). Therefore, we have

$$\frac{\partial \varphi}{\partial x^2} = F^2(x^1, x^2, 0) + \int\limits_0^{x_3} \frac{\partial F^3(x^1, x^2, x^3)}{\partial x^2}\, dx^3. \tag{11.6}$$

In order to transform the expression being integrated in (11.6) we use the formula (11.2). This leads to the following result:

$$\begin{aligned} \frac{\partial \varphi}{\partial x^2} &= F^2(x^1, x^2, 0) + \int\limits_0^{x_3} \frac{\partial F^2(x^1, x^2, x^3)}{\partial x^3}\, dx^3 = \\ &= F^2(x^1, x^2, 0) + F^2(x^1, x^2, x)\Big|_{x=0}^{x=x_3} = F^2(x^1, x^2, x^3). \end{aligned} \tag{11.7}$$

In calculating the derivative $\partial\varphi/\partial x^1$ we use that same tricks as in the case of the other two derivatives $\partial\varphi/\partial x^3$ and $\partial\varphi/\partial x^2$:

$$\begin{aligned}\frac{\partial\varphi}{\partial x^1}&=\frac{\partial}{\partial x^1}\int\limits_0^{x_1}F^1(x^1,0,0)\,dx^1+\frac{\partial}{\partial x^1}\int\limits_0^{x_2}F^2(x^1,x^2,0)\,dx^2+\\&+\frac{\partial}{\partial x^1}\int\limits_0^{x_3}F^3(x^1,x^2,x^3)\,dx^3=F^1(x^1,0,0)+\\&+\int\limits_0^{x_2}\frac{\partial F^2(x^1,x^2,0)}{\partial x^1}\,dx^2+\int\limits_0^{x_3}\frac{\partial F^3(x^1,x^2,x^3)}{\partial x^1}\,dx^3.\end{aligned}$$

To transform the last two integrals we use the relationships (11.1) and (11.3):

$$\begin{aligned}\frac{\partial\varphi}{\partial x^1}&=F^1(x^1,0,0)+F^1(x^1,x,0)\Bigr|_{x=0}^{x=x_2}+\\&\quad+F^1(x^1,x^2,x)\Bigr|_{x=0}^{x=x_3}=F^1(x^1,x^2,x^3).\end{aligned}\tag{11.8}$$

The relationships (11.5), (11.7), and (11.8) show that $\operatorname{grad}\varphi=\mathbf{F}$ for the function $\varphi(x^1,x^2,x^3)$ given by the formula (11.4). The theorem is proved. $\square$

THEOREM 11.2. *Any vorticular vector field $\mathbf{F}$ in the space $\mathbb{E}$ is the rotor of some other vector field $\mathbf{A}$, i.e. $\mathbf{F}=\operatorname{rot}\mathbf{A}$.*

PROOF. We perform the proof of this theorem in some rectangular Cartesian coordinate system with the orthonormal basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. The condition of vorticity for the field $\mathbf{F}$ in such a coordinate system is expressed by a single equation:

$$\frac{\partial F^1(\mathbf{x})}{\partial x^1}+\frac{\partial F^2(\mathbf{x})}{\partial x^2}+\frac{\partial F^3(\mathbf{x})}{\partial x^3}=0.\tag{11.9}$$

Let's construct the vector field $\mathbf{A}$ defining its components in the chosen coordinate system by the following three formulas:

$$\begin{aligned}A^1&=\int\limits_0^{x_3}F^2(x^1,x^2,x^3)\,dx^3-\int\limits_0^{x_2}F^3(x^1,x^2,0)\,dx^2,\\A^2&=-\int\limits_0^{x_3}F^1(x^1,x^2,x^3)\,dx^3,\\A^3&=0.\end{aligned}\tag{11.10}$$

Let's show that the field $\mathbf{A}$ with components (11.10) is that very field for which $\operatorname{rot}\mathbf{A}=\mathbf{F}$. We shall do it calculating directly the components of the rotor in the chosen coordinate system. For the first component we have

$$\frac{\partial A^3}{\partial x^2}-\frac{\partial A^2}{\partial x^3}=\frac{\partial}{\partial x^3}\int\limits_0^{x_3}F^1(x^1,x^2,x^3)\,dx^3=F^1(x^1,x^2,x^3).$$

Here we used the rule of differentiation of an integral with variable upper limit. In calculating the second component we take into account that the second integral in the expression for the component A^1 in (11.10) does not depend on x^3:

$$\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1} = \frac{\partial}{\partial x^3} \int\limits_0^{x_3} F^2(x^1, x^2, x^3)\, dx^3 = F^2(x^1, x^2, x^3).$$

And finally, for the third components of the rotor we derive

$$\begin{gathered}\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2} = -\int\limits_0^{x_3} \left(\frac{\partial F^1(x^1, x^2, x^3)}{\partial x^1} + \frac{\partial F^2(x^1, x^2, x^3)}{\partial x^2}\right) dx^3 + \\ + \frac{\partial}{\partial x^2} \int\limits_0^{x_2} F^3(x^1, x^2, 0)\, dx^2 = \int\limits_0^{x_3} \frac{\partial F^3(x^1, x^2, x^3)}{\partial x^3}\, dx^3 + \\ + F^3(x^1, x^2, 0) = F^3(x^1, x^2, x)\,\Big|_{x=0}^{x=x_3} + F^3(x^1, x^2, 0) = F^3(x^1, x^2, x^3).\end{gathered}$$

In these calculations we used the relationship (11.9) in order to replace the sum of two partial derivatives $\partial F^1/\partial x^1 + \partial F^2/\partial x^2$ by $-\partial F^3/\partial x^3$. Now, bringing together the results of calculating all three components of the rotor, we see that $\operatorname{rot} \mathbf{A} = \mathbf{F}$. Hence, the required field $\mathbf{A}$ can indeed be chosen in the form of (11.10). $\square$

CHAPTER III

CURVILINEAR COORDINATES

§ 1. Some examples of curvilinear coordinate systems.

The main purpose of Cartesian coordinate systems is the numeric representation of points: each point of the space $\mathbb{E}$ is represented by some unique triple of numbers (x^1, x^2, x^3). Curvilinear coordinate systems serve for the same purpose. We begin considering such coordinate systems with some examples.

Polar coordinates. Let's consider a plane, choose some point O on it (this will be the *pole*) and some ray OX coming out from this point. For an arbitrary point $A \neq O$ of that plane its position is determined by two parameters: the

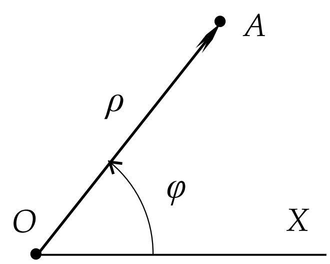

Fig. 1.1

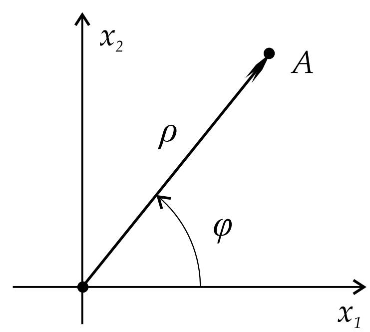

Fig. 1.2

length of its radius-vector $\rho = |\overrightarrow{OA}|$ and the value of the angle φ between the ray OX and the radius-vector of the point A. Certainly, one should also choose a positive (counterclockwise) direction to which the angle φ is laid (this is equivalent to choosing a preferable orientation on the plane). Angles laid to the opposite direction are understood as negative angles. The numbers ρ and φ are called the *polar coordinates* of the point A.

Let's associate some Cartesian coordinate system with the polar coordinates as shown on Fig. 1.2. We choose the point O as an origin, then direct the abscissa axis along the ray OX and get the ordinate axis from the abscissa axis rotating it by 90°. Then the Cartesian coordinates of the point A are derived from its polar coordinates by means of the formulas

$$\begin{cases} x^1 = \rho\,\cos(\varphi), \\ x^2 = \rho\,\sin(\varphi). \end{cases} \tag{1.1}$$

Conversely, one can express ρ and φ through x^1 and x^2 as follows:

$$\begin{cases} \rho = \sqrt{(x^1)^2 + (x^2)^2}, \\ \varphi = \arctan(x^2/x^1). \end{cases} \tag{1.2}$$

The pair of numbers (ρ, φ) can be treated as an element of the two-dimensional space $\mathbb{R}^2$. In order to express $\mathbb{R}^2$ visually we represent this space as a coordinate plane. The coordinate plane (ρ, φ) has not its own geometric interpretation, it is called the *map* of the polar coordinate system. Not all points of the map correspond to the real geometric points. The condition $\rho \geqslant 0$ excludes the whole half-plane of the map. The sine and cosine both are periodic functions with the period $2\pi = 360°$. Therefore there are different points of the map that represent the same geometric point. Thus, the mapping $(\rho, \varphi) \to (x^1, x^2)$ given by the formulas (1.1) is not injective. Let U be the unbounded domain highlighted with the light blue color on Fig. 1.3 (the points of the boundary are not included). Denote by V the image of the domain U under the mapping (1.1). It is easy to understand that V is the set of all points of the (x^1, x^2) plane except for those lying on the ray OX. If we restrict the mapping (1.1) to the domain U, we get the bijective mapping $m : U \to V$.

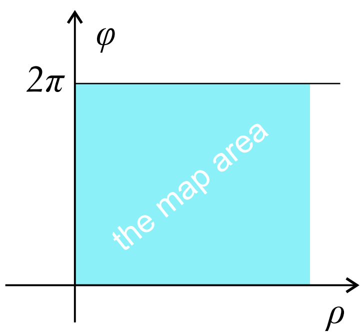

Fig. 1.3

Note that the formula (1.2) is not an exact expression for the inverse mapping $m^{-1}: V \to U$. The matter is that the values of $\tan(\varphi)$ at the points (x^1, x^2) and $(-x^1, -x^2)$ do coincide. In order to express m^{-1} exactly it would be better to use the tangent of the half angle:

$$\tan(\varphi/2) = \frac{x^2}{x^1 + \sqrt{(x^1)^2 + (x^2)^2}}.$$

However, we prefer the not absolutely exact expression for φ from (1.2) since it is relatively simple.

Let's draw the series of equidistant straight lines parallel to the axes on the map $\mathbb{R}^2$ of the polar coordinate system (see Fig. 1.4 below). The mapping (1.1) takes them to the series of rays and concentric circles on the (x^1, x^2) plane. The straight lines on Fig. 1.4 and the rays and circles on Fig. 1.5 compose the coordinate network of the polar coordinate system. By reducing the intervals between the lines one can obtain a more dense coordinate network. This procedure can be repeated infinitely many times producing more and more dense networks in each step. Ultimately (in the continuum limit), one can think the coordinate network to be maximally dense. Such a network consist of two families of lines: the first family is given by the condition $\varphi = \text{const}$, the second one — by the similar condition $\rho = \text{const}$.

On Fig. 1.4 exactly two coordinate lines pass through each point of the map: one is from the first family and the other is from the second family. On the (x^1, x^2) plane this condition is fulfilled at all points except for the origin O. Here all coordinate lines of the first family are crossed. The origin O is the only singular point of the polar coordinate system.

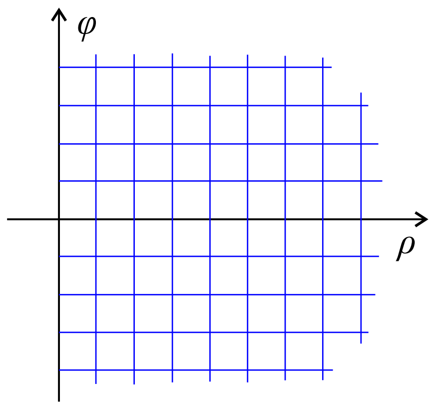

Fig. 1.4

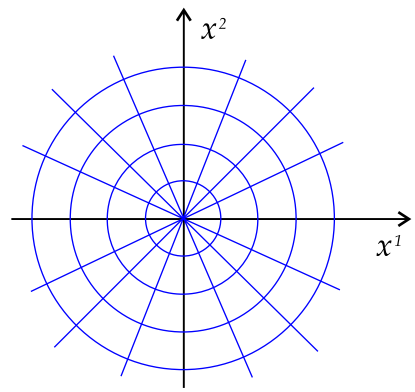

Fig. 1.5

The *cylindrical coordinate system* in the space $\mathbb{E}$ is obtained from the polar coordinates on a plane by adding the third coordinate h. As in the case of polar coordinate system, we associate some Cartesian coordinate system with the cylindrical coordinate system (see Fig. 1.6). Then

$$\begin{cases} x^1 = \rho\,\cos(\varphi), \\ x^2 = \rho\,\sin(\varphi), \\ x^3 = h. \end{cases} \tag{1.3}$$

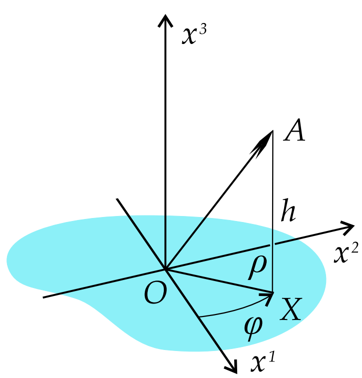

Fig. 1.6

Conversely, one can pass from Cartesian to cylindrical coordinates by means of the formula analogous to (1.2):

$$\begin{cases} \rho = \sqrt{(x^1)^2 + (x^2)^2}, \\ \varphi = \arctan(x^2/x^1), \\ h = x^3. \end{cases} \tag{1.4}$$

The coordinate network of the cylindrical coordinate system consists of three families of lines. These are the horizontal rays coming out from the points of the vertical axis Ox^3, the horizontal circles with the centers at the points of the axis Ox^3, and the vertical straight lines parallel to the axis Ox^3. The singular points of the cylindrical coordinate system fill the axis Ox^3. Exactly three coordinate lines (one from each family) pass through each regular point of the space $\mathbb{E}$, i. e. through each point that does not lie on the axis Ox^3.

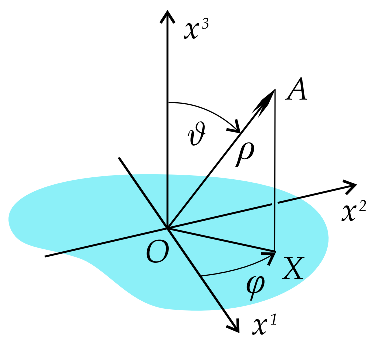

Fig. 1.7

The *spherical coordinate system* in the space $\mathbb{E}$ is obtained by slight modification of the cylindrical coordinates. The coordinate h is replaced by the angular coordinate ϑ, while the quantity ρ in spherical coordinates denotes the length of the radius-vector of the point A (see Fig. 1.7). Then

$$\begin{cases} x^1 = \rho\,\sin(\vartheta)\,\cos(\varphi), \\ x^2 = \rho\,\sin(\vartheta)\,\sin(\varphi), \\ x^3 = \rho\,\cos(\vartheta). \end{cases} \tag{1.5}$$

The spherical coordinates of a point are usually written in the following order: ρ is the first coordinate, ϑ is the second one, and φ is the third coordinate. The converse transition from Cartesian coordinates to these quantities is given by the formula:

$$\begin{cases} \rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \\ \vartheta = \arccos\left(x^3/\sqrt{((x^1)^2 + (x^2)^2 + (x^3)^2}\right), \\ \varphi = \arctan(x^2/x^1). \end{cases} \tag{1.6}$$

Coordinate lines of spherical coordinates form three families. The first family is composed of the rays coming out from the point O; the second family is formed by circles that lie in various vertical planes passing through the axix Ox^3; and the third family consists of horizontal circles whose centers are on the axis Ox^3. Exactly three coordinate lines pass through each regular point of the space $\mathbb{E}$, one line from each family.

The condition $\rho = \text{const}$ specifies the sphere of the radius ρ in the space $\mathbb{E}$. The coordinate lines of the second and third families define the network of meridians and parallels on this sphere exactly the same as used in geography to define the coordinates on the Earth surface.

§ 2. Moving frame of a curvilinear coordinate system.

Let D be some domain in the space $\mathbb{E}$. Saying domain, one usually understand a connected open set. An open set D means that along with each its point $A \in D$ the set D comprises some spherical neighborhood $O(A)$ of this point. A connected set D means that any two points of this set can be connected by a smooth curve lying within D. See more details in [2]. Let's consider three numeric functions $u^1(\mathbf{x})$, $u^2(\mathbf{x})$, and $u^3(\mathbf{x})$ defined in the domain D. Generally speaking, their domains could be wider, but we need them only within D. The values of three functions u^1, u^2, u^3 at each point form a triple of numbers, they can be interpreted as a point of the space $\mathbb{R}^3$. Then the triple of functions u^1, u^2, u^3 define a mapping $\mathbf{u}\colon D \to \mathbb{R}^3$.

Definition 2.1. A triple of differentiable functions u^1, u^2, u^3 is called *regular at a point* A of the space $\mathbb{E}$ if the gradients of these functions $\operatorname{grad} u^1$, $\operatorname{grad} u^2$, and $\operatorname{grad} u^3$ are linearly independent at the point A.

Let's choose some Cartesian coordinate system in $\mathbb{E}$, in this coordinate system the above functions u^1, u^2, u^3 are represented by the functions $u^i = u^i(x^1, x^2, x^3)$ of Cartesian coordinates of a point. The gradients of the differentiable functions u^1, u^2, u^3 form the triple of covectorial fields whose components are given by the partial derivatives of u^1, u^2, u^3 with respect to x^1, x^2, x^3:

$$\operatorname{grad} u^i = \left(\frac{\partial u^i}{\partial x^1}, \frac{\partial u^i}{\partial x^2}, \frac{\partial u^i}{\partial x^3}\right). \tag{2.1}$$

Let's compose a matrix of the gradients (2.1):

$$J = \begin{Vmatrix} \dfrac{\partial u^1}{\partial x^1} & \dfrac{\partial u^1}{\partial x^2} & \dfrac{\partial u^1}{\partial x^3} \\ \dfrac{\partial u^2}{\partial x^1} & \dfrac{\partial u^2}{\partial x^2} & \dfrac{\partial u^2}{\partial x^3} \\ \dfrac{\partial u^3}{\partial x^1} & \dfrac{\partial u^3}{\partial x^2} & \dfrac{\partial u^3}{\partial x^3} \end{Vmatrix}. \tag{2.2}$$

The matrix J of the form (2.2) is called the *Jacobi matrix* of the mapping $\mathbf{u}\colon D \to R^3$ given by the triple of the differentiable functions u^1, u^2, u^3 in the domain D. It is obvious that the regularity of the functions u^1, u^2, u^3 at a point is equivalent to the non-degeneracy of the Jacobi matrix at that point: $\det J \neq 0$.

Theorem 2.1. *If continuously differentiable functions u^1, u^2, u^3 with the domain D are regular at a point A, then there exists some neighborhood $O(A)$ of the point A and some neighborhood $O(\mathbf{u}(A))$ of the point $\mathbf{u}(A)$ in the space $\mathbb{R}^3$ such that the following conditions are fulfilled:*

(1) *the mapping $\mathbf{u}\colon O(A) \to O(\mathbf{u}(A))$ is bijective;*
(2) *the inverse mapping $\mathbf{u}^{-1}\colon O(\mathbf{u}(A)) \to O(A)$ is continuously differentiable.*

The theorem 2.1 or propositions equivalent to it are usually proved in the course of mathematical analysis (see [2]). They are known as the theorems on implicit functions.

Definition 2.2. Say that an ordered triple of continuously differentiable functions u^1, u^2, u^3 with the domain $D \subset \mathbb{E}$ define a *curvilinear coordinate system* in D if it is regular at all points of D and if the mapping $\mathbf{u}$ determined by them is a bijective mapping from D to some domain $U \subset \mathbb{R}^3$.

The cylindrical coordinate system is given by three functions $u^1 = \rho(\mathbf{x})$, $u^2 = \varphi(\mathbf{x})$, and $u^3 = h(\mathbf{x})$ from (1.4), while the spherical coordinate system is given by the functions (1.6). However, the triples of functions (1.4) and (1.6) satisfy the conditions from the definition 2.2 only after reducing somewhat their domains. Upon proper choice of a domain D for (1.4) and (1.6) the inverse mappings $\mathbf{u}^{-1}$ are given by the formulas (1.3) and (1.5).

Suppose that in a domain $D \subset \mathbb{E}$ a curvilinear coordinate system u^1, u^2, u^3 is given. Let's choose an auxiliary Cartesian coordinate system in $\mathbb{E}$. Then u^1, u^2, u^3 is a triple of functions defining a map $\mathbf{u}$ from D onto some domain $U \subset \mathbb{R}^3$:

$$\begin{cases} u^1 = u^1(x^1, x^2, x^3), \\ u^2 = u^2(x^1, x^2, x^3), \\ u^3 = u^3(x^1, x^2, x^3). \end{cases} \tag{2.3}$$

The domain D is called the *domain being mapped*, the domain $U \subset \mathbb{R}^3$ is called the *map* or the *chart*, while $\mathbf{u}^{-1}\colon U \to D$ is called the *chart mapping*. The chart mapping is given by the following three functions:

$$\begin{cases} x^1 = x^1(u^1, u^2, u^3),\\ x^2 = x^2(u^1, u^2, u^3),\\ x^3 = x^3(u^1, u^2, u^3). \end{cases} \tag{2.4}$$

Denote by $\mathbf{r}$ the radius-vector $\mathbf{r}$ of the point with Cartesian coordinates x^1, x^2, x^3. Then instead of three scalar functions (2.4) we can use one vectorial function

$$\mathbf{r}(u^1, u^2, u^3) = \sum_{q=1}^{3} x^q(u^1, u^2, u^3) \cdot \mathbf{e}_q. \tag{2.5}$$

Let's fix some two of three coordinates u^1, u^2, u^3 and let's vary the third of them. Thus we get three families of straight lines within the domain $U \subset \mathbb{R}^3$:

$$\begin{cases} u^1 = t,\\ u^2 = c^2,\\ u^3 = c^3, \end{cases} \qquad \begin{cases} u^1 = c^1,\\ u^2 = t,\\ u^3 = c^3, \end{cases} \qquad \begin{cases} u^1 = c^1,\\ u^2 = c^2,\\ u^3 = t. \end{cases} \tag{2.6}$$

Here c^1, c^2, c^3 are constants. The straight lines (2.6) form a rectangular coordinate network within the chart U. Exactly one straight line from each of the families (2.6) passes through each point of the chart. Substituting (2.6) into (2.5) we map the rectangular network from U onto a curvilinear network in the domain $D \subset E$. Such a network is called the *coordinate network* of a curvilinear coordinate system.

The coordinate network of a curvilinear coordinate system on the domain D consists of three families of lines. Due to the bijectivity of the mapping $\mathbf{u}\colon D \to U$ exactly three coordinate lines pass through each point of the domain D — one line from each family. Each coordinate line has its canonical parametrization: $t = u^1$ is the parameter for the lines of the first family, $t = u^2$ is the parameter for the lines of the second family, and finally, $t = u^3$ is the parameter for the lines of the third family. At each point of the domain D we have three tangent vectors, they are tangent to the coordinate lines of the three families passing through that point. Let's denote them $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. The vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are obtained by differentiating the radius-vector $\mathbf{r}(u^1, u^2, u^3)$ with respect to the parameters u^1, u^2, u^3 of coordinate lines. Therefore, we can write

$$\mathbf{E}_j(u^1, u^2, u^3) = \frac{\partial\, \mathbf{r}(u^1, u^2, u^3)}{\partial u^j}. \tag{2.7}$$

Let's substitute (2.5) into (2.7). The basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ do not depend on the variables u^1, u^2, u^3, hence, we get

$$\mathbf{E}_j(u^1, u^2, u^3) = \sum_{q=1}^{3} \frac{\partial x^q(u^1, u^2, u^3)}{\partial u^j} \cdot \mathbf{e}_q. \tag{2.8}$$

The formula (2.8) determines the expansion of the vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The column-vectors of the coordinates of $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ can be concatenated into the following matrix:

$$I = \begin{Vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^1}{\partial u^3} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \dfrac{\partial x^2}{\partial u^3} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} & \dfrac{\partial x^3}{\partial u^3} \end{Vmatrix}. \tag{2.9}$$

Comparing (2.9) and (2.2), we see that (2.9) is the Jacobi matrix for the mapping $\mathbf{u}^{-1}\colon U \to D$ given by the functions (2.4). Let's substitute (2.4) into (2.3):

$$u^i(x^1(u^1, u^2, u^3), x^2(u^1, u^2, u^3), x^3(u^1, u^2, u^3)) = u^i. \tag{2.10}$$

The identity (2.10) follows from the fact that the functions (2.3) and (2.4) define two mutually inverse mappings $\mathbf{u}$ and $\mathbf{u}^{-1}$. Let's differentiate the identity (2.10) with respect to the variable u^j:

$$\sum_{q=1}^{3} \frac{\partial u^i(x^1, x^2, x^3)}{\partial x^q} \, \frac{\partial x^q(u^1, u^2, u^3)}{\partial u^j} = \delta^i_j. \tag{2.11}$$

Here we used the chain rule for differentiating the composite function in (2.10). The relationship (2.11) shows that the matrices (2.2) and (2.9) are inverse to each other. More precisely, we have the following relationship

$$I(u^1, u^2, u^3) = J(x^1, x^2, x^3)^{-1}, \tag{2.12}$$

where x^1, x^2, x^3 should be expressed through u^1, u^2, u^3 by means of (2.4), or conversely, u^1, u^2, u^3 should be expressed through x^1, x^2, x^3 by means of (2.3). The arguments shown in the relationship (2.12) are the natural arguments for the components of the Jacobi matrices I and J. However, one can pass to any required set of variables by means of (2.3) or (2.4) whenever it is necessary.

The regularity of the triple of functions (2.3) defining a curvilinear coordinate system in the domain D means that the matrix (2.2) is non-degenerate. Then, due to (2.12), the inverse matrix (2.9) is also non-degenerate. Therefore, the vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ given by the formula (2.8) are linearly independent at any point of the domain D. Due to the linear independence of the *coordinate tangent vectors* $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ they form a *moving frame* which is usually called the *coordinate frame* of the curvilinear coordinate system. The formula (2.8) now can be interpreted as the transition formula for passing from the basis of the auxiliary Cartesian coordinate system to the basis formed by the vectors of the frame $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$:

$$\mathbf{E}_j = \sum_{q=1}^{3} S^q_j(u^1, u^2, u^3) \cdot \mathbf{e}_q. \tag{2.13}$$

The transition matrix S in the formula (2.13) coincides with the Jacobi matrix (2.9), therefore its components depend on u^1, u^2, u^3. These are the natural variables for the components of S.

The inverse transition from the basis $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ to the basis of the Cartesian coordinate system is given by the inverse matrix $T = S^{-1}$. Due to (2.12) the inverse transition matrix coincides with the Jacobi matrix (2.2). Therefore, x^1, x^2, x^3 are the natural variables for the components of the matrix T:

$$\mathbf{e}_q = \sum_{i=1}^{3} T^i_q(x^1, x^2, x^3) \cdot \mathbf{E}_i. \tag{2.14}$$

The vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ of the moving frame depend on a point of the domain $D \subset E$. Since in a curvilinear coordinate system such a point is represented by its coordinates u^1, u^2, u^3, these variables are that very arguments which are natural for the vectors of the moving frame: $\mathbf{E}_i = \mathbf{E}_i(u^1, u^2, u^3)$.

§ 3. Change of curvilinear coordinates.

Let u^1, u^2, u^3 be some curvilinear coordinates in some domain D_1 and let $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ be some other curvilinear coordinates in some other domain D_2. If the domains D_1 and D_2 do intersect, then in the domain $D = D_1 \cap \tilde{D}_2$ we have two coordinate systems. We denote by U and $\tilde{U}$ the preimages of the domain

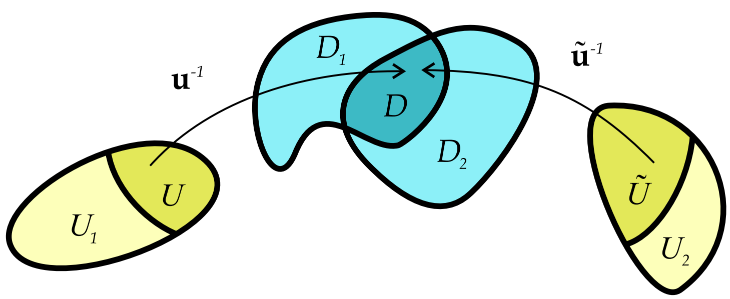

Fig. 3.1

D in the maps U_1 and U_2, i.e. we denote $U = \mathbf{u}(D_1 \cap \tilde{D}_2)$ and we denote $\tilde{U} = \tilde{\mathbf{u}}(D_1 \cap \tilde{D}_2)$. Due to the chart mappings the points of the domain D are in one-to-one correspondence with the points of the domains U and $\tilde{U}$. As for the chart mappings $\mathbf{u}^{-1}$ and $\tilde{\mathbf{u}}^{-1}$, they are given by the following functions:

$$\begin{cases} x^1 = x^1(u^1, u^2, u^3), \\ x^2 = x^2(u^1, u^2, u^3), \\ x^3 = x^3(u^1, u^2, u^3), \end{cases} \qquad \begin{cases} x^1 = x^1(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3), \\ x^2 = x^2(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3), \\ x^3 = x^3(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \end{cases} \tag{3.1}$$

The mappings $\mathbf{u}$ and $\tilde{\mathbf{u}}$ inverse to the chart mappings are given similarly:

$$\begin{cases} u^1 = u^1(x^1, x^2, x^3), \\ u^2 = u^2(x^1, x^2, x^3), \\ u^3 = u^3(x^1, x^2, x^3), \end{cases} \qquad \begin{cases} \tilde{u}^1 = \tilde{u}^1(x^1, x^2, x^3), \\ \tilde{u}^2 = \tilde{u}^2(x^1, x^2, x^3), \\ \tilde{u}^3 = \tilde{u}^3(x^1, x^2, x^3). \end{cases} \tag{3.2}$$

Let's substitute the first set of functions (3.1) into the arguments of the second set of functions (3.2). Similarly, we substitute the second set of functions (3.1) into the arguments of the first set of functions in (3.2). As a result we get the functions

$$\begin{cases} \tilde{u}^1(x^1(u^1,u^2,u^3), x^2(u^1,u^2,u^3), x^3(u^1,u^2,u^3)),\\ \tilde{u}^2(x^1(u^1,u^2,u^3), x^2(u^1,u^2,u^3), x^3(u^1,u^2,u^3)),\\ \tilde{u}^3(x^1(u^1,u^2,u^3), x^2(u^1,u^2,u^3), x^3(u^1,u^2,u^3)),\end{cases} \tag{3.3}$$

$$\begin{cases} u^1(x^1(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3), x^2(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3), x^3(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3)),\\ u^2(x^1(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3), x^2(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3), x^3(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3)),\\ u^3(x^1(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3), x^2(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3), x^3(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3))\end{cases} \tag{3.4}$$

which define the pair of mutually inverse mappings $\tilde{\mathbf{u}}\circ\mathbf{u}^{-1}$ and $\mathbf{u}\circ\tilde{\mathbf{u}}^{-1}$. For the sake of brevity we write these sets of functions as follows:

$$\begin{cases} \tilde{u}^1 = \tilde{u}^1(u^1,u^2,u^3),\\ \tilde{u}^2 = \tilde{u}^2(u^1,u^2,u^3),\\ \tilde{u}^3 = \tilde{u}^3(u^1,u^2,u^3),\end{cases} \qquad \begin{cases} u^1 = u^1(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3),\\ u^2 = u^2(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3),\\ u^3 = u^3(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3).\end{cases} \tag{3.5}$$

The formulas (3.5) express the coordinates of a point from the domain D in some curvilinear coordinate system through its coordinates in some other coordinate system. These formulas are called the *transformation formulas* or the formulas for *changing the curvilinear coordinates*.

Each of the two curvilinear coordinate systems has its own moving frame within the domain $D = D_1 \cap D_2$. Let's denote by S and T the transition matrices relating these two moving frames. Then we can write

$$\tilde{\mathbf{E}}_j = \sum_{i=1}^{3} S^i_j \cdot \mathbf{E}_i, \qquad \mathbf{E}_i = \sum_{k=1}^{3} T^k_i \cdot \tilde{\mathbf{E}}_k. \tag{3.6}$$

Theorem 3.1. *The components of the transition matrices S and T for the moving frames of two curvilinear coordinate system in (3.6) are determined by the partial derivatives of the functions (3.5):*

$$S^i_j(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3) = \frac{\partial u^i}{\partial \tilde{u}^j}, \qquad T^k_i(u^1,u^2,u^3) = \frac{\partial \tilde{u}^k}{\partial u^i}. \tag{3.7}$$

Proof. We shall prove only the first formula in (3.7). The proof of the second formula is absolutely analogous to the proof of the first one. Let's choose some auxiliary Cartesian coordinate system and then write the formula (2.8) applied to the frame vectors of the second curvilinear coordinate system:

$$\tilde{\mathbf{E}}_j(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3) = \sum_{q=1}^{3} \frac{\partial x^q(\tilde{u}^1,\tilde{u}^2,\tilde{u}^3)}{\partial \tilde{u}^j} \cdot \mathbf{e}_q. \tag{3.8}$$

Applying the formula (2.14), we express $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ through $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. Remember that the matrix T in (2.14) coincides with the Jacobi matrix $J(x^1,x^2,x^3)$ from

(2.2). Therefore, we can write the following formula:

$$\mathbf{e}_q = \sum_{i=1}^{3} \frac{\partial u^i(x^1, x^2, x^3)}{\partial x^q} \cdot \mathbf{E}_i. \tag{3.9}$$

Now let's substitute (3.9) into (3.8). As a result we get the formula relating the frame vectors of two curvilinear coordinate systems:

$$\tilde{\mathbf{E}}_j = \sum_{i=1}^{3} \left(\sum_{q=1}^{3} \frac{\partial u^i(x^1, x^2, x^3)}{\partial x^q} \, \frac{\partial x^q(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)}{\partial \tilde{u}^j} \right) \cdot \mathbf{E}_i. \tag{3.10}$$

Comparing (3.10) and (3.6), from this comparison for the components of S we get

$$S^i_j = \sum_{q=1}^{3} \frac{\partial u^i(x^1, x^2, x^3)}{\partial x^q} \, \frac{\partial x^q(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)}{\partial \tilde{u}^j}. \tag{3.11}$$

Remember that the Cartesian coordinates x^1, x^2, x^3 in the above formula (3.11) are related to the curvilinear coordinates $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ by means of (3.1). Hence, the sum in right hand side of (3.11) can be transformed to the partial derivative of the composite function $u^i((x^1(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3), x^2(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3), x^3(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3))$ from (3.4):

$$S^i_j = \frac{\partial u^i}{\partial \tilde{u}^j}.$$

Note that the functions (3.4) written in the form of (3.5) are that very functions relating $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ and u^1, u^2, u^3, and their derivatives are in formula (3.7). The theorem is proved. $\square$

A remark on the orientation. From the definition 2.2 we derive that the functions (2.3) are continuously differentiable. Due to the theorem 2.1 the functions (2.4) representing the inverse mappings are also continuously differentiable. Then the components of the matrix S in the formula (2.13) coinciding with the components of the Jacobi matrix (2.9) are continuous functions within the domain U. The same is true for the determinant of the matrix S: the determinant $\det S(u^1, u^2, u^3)$ is a continuous function in the domain U which is nonzero at all points of this domain. A nonzero continuous real function in a connected set U cannot take the values of different signs in U. This means that $\det S > 0$ or $\det S < 0$. This means that the orientation of the triple of vectors forming the moving frame of a curvilinear coordinate system is the same for all points of a domain where it is defined. Since the space $\mathbb{E}$ is equipped with the preferable orientation, we can subdivide all curvilinear coordinates in $\mathbb{E}$ into right-oriented and left-oriented coordinate systems.

A remark on the smoothness. The definition 2.2 yields the concept of a continuously differentiable curvilinear coordinate system. However, the functions (2.3) could belong to a higher smoothness class C^m. In this case we say that we have a *curvilinear coordinate system of the smoothness class* C^m. The components of the Jacobi matrix (2.2) for such a coordinate system are the functions of the class C^{m-1}. Due to the relationship (2.12) the components of the Jacobi matrix

(2.9) belong to the same smoothness class C^{m-1}. Hence, the functions (2.4) belong to the smoothness class C^m.

If we have two curvilinear coordinate systems of the smoothness class C^m, then, according to the above considerations, the transformation functions (3.5) belong to the class C^m, while the components of the transition matrices S and T given by the formulas (3.7) belong to the smoothness class C^{m-1}.

§ 4. Vectorial and tensorial fields in curvilinear coordinates.

Let u^1, u^2, u^3 be some curvilinear coordinate system in some domain $D \subset \mathbb{E}$ and let $\mathbf{F}$ be some vector field defined at the points of the domain D. Then at a point with coordinates u^1, u^2, u^3 we have the field vector $\mathbf{F}(u^1, u^2, u^3)$ and the triple of the frame vectors $\mathbf{E}_1(u^1, u^2, u^3)$, $\mathbf{E}_2(u^1, u^2, u^3)$, $\mathbf{E}_3(u^1, u^2, u^3)$. Let's expand the field vector $\mathbf{F}$ in the basis formed by the frame vectors:

$$\mathbf{F}(u^1, u^2, u^3) = \sum_{i=1}^{3} F^i(u^1, u^2, u^3) \cdot \mathbf{E}_i(u^1, u^2, u^3). \tag{4.1}$$

The quantities $F^i(u^1, u^2, u^3)$ in such expansion are naturally called the *components* of the vector field $\mathbf{F}$ in the curvilinear coordinates u^1, u^2, u^3. If we have another curvilinear coordinate system $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ in the domain D, then we have the other expansion of the form (4.1):

$$\mathbf{F}(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3) = \sum_{i=1}^{3} \tilde{F}^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3) \cdot \tilde{\mathbf{E}}_i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \tag{4.2}$$

By means of the formulas (3.6) one can easily derive the relationships binding the components of the field $\mathbf{F}$ in the expansions (4.1) and (4.2):

$$\begin{aligned} &F^i(\mathbf{u}) = \sum_{j=1}^{3} S^i_j(\tilde{\mathbf{u}})\ \tilde{F}^j(\tilde{\mathbf{u}}),\\ &u^i = u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \end{aligned} \tag{4.3}$$

The relationships (4.3) are naturally interpreted as the generalizations for the relationships (1.2) from Chapter II for the case of curvilinear coordinates.

Note that Cartesian coordinate systems can be treated as a special case of curvilinear coordinates. The transition functions $u^i = u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)$ in the case of a pair of Cartesian coordinate systems are linear, therefore the matrix S calculated according to the theorem 3.1 in this case is a constant matrix.

Now let $\mathbf{F}$ be either a field of covectors, a field of linear operators, or a field of bilinear forms. In any case the components of the field F at some point are determined by fixing some basis attached to that point. The vectors of the moving frame of a curvilinear coordinate system at a point with coordinates u^1, u^2, u^3 provide the required basis. The components of the field $\mathbf{F}$ determined by this basis are called the *components* of the field $\mathbf{F}$ in that curvilinear coordinates. The transformation rules for the components of the fields listed above under a change of curvilinear coordinates generalize the formulas (1.3), (1.4), and (1.5) from

Chapter II. For a covectorial field $\mathbf{F}$ the transformation rule for its components under a change of coordinates looks like

$$\begin{aligned} &F_i(\mathbf{u}) = \sum_{j=1}^{3} T_i^j(\mathbf{u})\ \tilde{F}_j(\tilde{\mathbf{u}}), \\ &u^i = u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \end{aligned} \tag{4.4}$$

The transformation rule for the components of an operator field $\mathbf{F}$ is written as

$$\begin{aligned} &F_j^i(\mathbf{x}) = \sum_{p=1}^{3}\sum_{q=1}^{3} S_p^i(\tilde{\mathbf{u}})\, T_j^q(\mathbf{u})\ \tilde{F}_q^p(\tilde{\mathbf{x}}), \\ &u^i = u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \end{aligned} \tag{4.5}$$

In the case of a field of bilinear (quadratic) forms the generalization of the formula (1.5) from Chapter II looks like

$$\begin{aligned} &F_{ij}(\mathbf{u}) = \sum_{p=1}^{3}\sum_{q=1}^{3} T_i^p(\mathbf{u})\, T_j^q(\mathbf{u})\ \tilde{F}_{pq}(\tilde{\mathbf{u}}), \\ &u^i = u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \end{aligned} \tag{4.6}$$

Let $\mathbf{F}$ be a tensor field of the type (r, s). In contrast to a vectorial field, the value of such a tensorial field at a point have no visual embodiment in form of an arrowhead segment. Moreover, in general case there is no visually explicit way of finding the numerical values for the components of such a field in a given basis. However, according to the definition 1.1 from Chapter II, a tensor is a geometric object that for each basis has an array of components associated with this basis. Let's denote by $\mathbf{F}(u^1, u^2, u^3)$ the value of the field $\mathbf{F}$ at the point with coordinates u^1, u^2, u^3. This is a tensor whose components in the basis $\mathbf{E}_1(u^1, u^2, u^3)$, $\mathbf{E}_2(u^1, u^2, u^3)$, $\mathbf{E}_3(u^1, u^2, u^3)$ are called components of the field $\mathbf{F}$ in a given curvilinear coordinate system. The transformation rules for the components of a tensor field under a change of a coordinate system follow from the formula (1.6) in Chapter II. For a tensorial field of the type (r, s) it looks like

$$\begin{aligned} &F^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}(\mathbf{u}) = \sum_{\substack{p_1\ldots\, p_r\\ q_1\ldots\, q_s}} S^{i_1}_{p_1}(\tilde{\mathbf{u}}) \ldots S^{i_r}_{p_r}(\tilde{\mathbf{u}}) \times \\ &\qquad\qquad \times T^{q_1}_{j_1}(\mathbf{u}) \ldots T^{q_s}_{j_s}(\mathbf{u})\ \tilde{F}^{p_1\ldots\, p_r}_{q_1\ldots\, q_s}(\tilde{\mathbf{u}}), \\ &u^i = u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3). \end{aligned} \tag{4.7}$$

The formula (4.7) has two important differences as compared to the corresponding formula (1.7) in Chapter II. In the case of curvilinear coordinates

(1) the transition functions $u^i(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)$ should not be linear functions;
(2) the transition matrices $S(\tilde{\mathbf{u}})$ and $T(\mathbf{u})$ are not necessarily constant matrices.

Note that these differences do not affect the algebraic operations with tensorial fields. The operations of addition, tensor product, contraction, index permutation, symmetrization, and alternation are implemented by the same formulas as in

Cartesian coordinates. The differences (1) and (2) reveal only in the operation of covariant differentiation of tensor fields.

Any curvilinear coordinate system is naturally equipped with the the metric tensor $\mathbf{g}$. This is a tensor whose components are given by mutual scalar products of the frame vectors for a given coordinate system:

$$g_{ij} = (\mathbf{E}_i(\mathbf{u}) \,|\, \mathbf{E}_j(\mathbf{u})). \tag{4.8}$$

The components of the inverse metric tensor $\hat{\mathbf{g}}$ are obtained by inverting the matrix $\mathbf{g}$. In a curvilinear coordinates the quantities g_{ij} and g^{ij} are not necessarily constants any more.

We already know that the metric tensor $\mathbf{g}$ defines the volume pseudotensor $\boldsymbol{\omega}$. As before, in curvilinear coordinates its components are given by the formula (6.11) from Chapter II. Since the space $\mathbb{E}$ has the preferable orientation, the volume pseudotensor can be transformed to the volume tensor $\boldsymbol{\omega}$. The formula (8.1) from Chapter II for the components of this tensor remains valid in a curvilinear coordinate system either.

§ 5. Differentiation of tensor fields in curvilinear coordinates.

Let $\mathbf{A}$ be a differentiable tensor field of the type (r, s). In § 5 of Chapter II we have defined the concept of covariant differential. The covariant differential $\nabla\mathbf{A}$ of a field $\mathbf{A}$ is a tensorial field of the type $(r, s+1)$. In an arbitrary Cartesian coordinate system the components of the field $\nabla\mathbf{A}$ are obtained by differentiating the components of the original field $\mathbf{A}$ with respect to x^1, x^2, and x^3. The use of curvilinear coordinates does not annul the operation of covariant differentiation. However, the procedure of deriving the components of the field $\nabla\mathbf{A}$ from the components of $\mathbf{A}$ in curvilinear coordinates is more complicated.

Let u^1, u^2, u^3 be some curvilinear coordinate system in a domain $D \subset \mathbb{E}$. Let's derive the rule for covariant differentiation of tensor fields in a curvilinear coordinate system. We consider a vectorial field $\mathbf{A}$ to begin with. This is a field whose components are specified by one upper index: $A^i(u^1, u^2, u^3)$. In order to calculate the components of the field $\mathbf{B} = \nabla\mathbf{A}$ we choose some auxiliary Cartesian coordinate system $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$. Then we need to do the following maneuver: first we transform the components of $\mathbf{A}$ from curvilinear coordinates to Cartesian ones, then calculate the components of the field $\mathbf{B} = \nabla\mathbf{A}$ by means of the formula (5.1) from Chapter II, and finally, we transform the components of $\nabla\mathbf{A}$ from Cartesian coordinates back to the original curvilinear coordinates.

The Cartesian coordinates $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$ and the curvilinear coordinates u^1, u^2, u^3 are related by the following transition functions:

$$\begin{cases} \tilde{x}^1 = \tilde{x}^1(u^1, u^2, u^3), \\ \tilde{x}^2 = \tilde{x}^2(u^1, u^2, u^3), \\ \tilde{x}^3 = \tilde{x}^3(u^1, u^2, u^3), \end{cases} \qquad \begin{cases} u^1 = u^1(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \\ u^2 = u^2(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \\ u^3 = u^3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3). \end{cases} \tag{5.1}$$

The components of the corresponding transition matrices are calculated according to the formula (3.7). When applied to (5.1), this formula yields

$$S^i_j(\tilde{\mathbf{x}}) = \frac{\partial u^i}{\partial \tilde{x}^j}, \qquad T^k_i(\mathbf{u}) = \frac{\partial \tilde{x}^k}{\partial u^i}. \tag{5.2}$$

Denote by $\tilde{A}^k(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ the components of the vector field $\mathbf{A}$ in the Cartesian coordinate system $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$. Then we get

$$\tilde{A}^k = \sum_{p=1}^{3} T^k_p(\mathbf{u})\, A^p(\mathbf{u}).$$

For the components of the field $\mathbf{B} = \nabla\mathbf{A}$ in these Cartesian coordinates, applying the formula (5.1) from Chapter II, we get

$$\tilde{B}^k_q = \frac{\partial \tilde{A}^k}{\partial \tilde{x}^q} = \sum_{p=1}^{3} \frac{\partial}{\partial \tilde{x}^q}\left(T^k_p(\mathbf{u})\, A^p(\mathbf{u})\right). \tag{5.3}$$

Now we perform the inverse transformation of the components of $\mathbf{B}$ from the Cartesian coordinates $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$ back to the curvilinear coordinates u^1, u^2, u^3:

$$\nabla_j A^i = B^i_j(\mathbf{u}) = \sum_{k=1}^{3}\sum_{q=1}^{3} S^i_k(\tilde{\mathbf{x}})\, T^q_j(\mathbf{u})\, \tilde{B}^k_q. \tag{5.4}$$

Let's apply the Leibniz rule for calculating the partial derivative in (5.3). As a result we get two sums. Then, substituting these sums into (5.4), we obtain

$$\begin{aligned}\nabla_j A^i = \sum_{q=1}^{3}\sum_{p=1}^{3}\left(\sum_{k=1}^{3} S^i_k(\tilde{\mathbf{x}})\, T^k_p(\mathbf{u})\right) T^q_j(\mathbf{u})\,\frac{\partial A^p(\mathbf{u})}{\partial \tilde{x}^q}+\\ +\sum_{p=1}^{3}\left(\sum_{q=1}^{3}\sum_{p=1}^{3} S^i_k(\tilde{\mathbf{x}})\, T^q_j(\mathbf{u})\,\frac{\partial T^k_p(\mathbf{u})}{\partial \tilde{x}^q}\right) A^p(\mathbf{u}).\end{aligned}$$

Note that the matrices S and T are inverse to each other. Therefore, we can calculate the sums over k and p in the first summand. Moreover, we replace $T^q_j(\mathbf{u})$ by the derivatives $\partial\tilde{x}^q/\partial u^j$ due to the formula (5.2), and we get

$$\sum_{q=1}^{3} T^q_j(\mathbf{u})\,\frac{\partial}{\partial \tilde{x}^q} = \sum_{q=1}^{3}\frac{\partial \tilde{x}^q}{\partial u^j}\,\frac{\partial}{\partial \tilde{x}^q} = \frac{\partial}{\partial u^j}.$$

Taking into account all the above arguments, we transform the formula for the covariant derivative $\nabla_j A^i$ into the following one:

$$\nabla_j A^i(\mathbf{u}) = \frac{\partial A^i(\mathbf{u})}{\partial u^j} + \sum_{p=1}^{3}\left(\sum_{k=1}^{3} S^i_k(\tilde{\mathbf{x}})\,\frac{\partial T^k_p(\mathbf{u})}{\partial u^j}\right) A^p(\mathbf{u}).$$

We introduce the special notation for the sum enclosed into the round brackets in the above formula, we denote it by Γ^i_{jp}:

$$\Gamma^i_{jp}(\mathbf{u}) = \sum_{k=1}^{3} S^i_k(\tilde{\mathbf{x}})\,\frac{\partial T^k_p(\mathbf{u})}{\partial u^j}. \tag{5.5}$$

Taking into account the notations (5.5), now we can write the rule of covariant differentiation of a vector field in curvilinear coordinates as follows:

$$\nabla_{j} A^{i}=\frac{\partial A^{i}}{\partial u^{j}}+\sum_{p=1}^{3} \Gamma_{j p}^{i} A^{p}. \tag{5.6}$$

The quantities Γ^i_{jp} calculated according to (5.5) are called the *connection components* or the *Christoffel symbols*. These quantities are some inner characteristics of a curvilinear coordinate system. This fact is supported by the following lemma.

LEMMA 5.1. *The connection components Γ^i_{jp} of a curvilinear coordinate system u^1, u^2, u^3 given by the formula (5.5) do not depend on the choice of an auxiliary Cartesian coordinate system $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$.*

PROOF. Let's multiply both sides of the equality (5.5) by the frame vector $\mathbf{E}_i$ and then sum over the index i:

$$\sum_{i=1}^{3} \Gamma_{j p}^{i}(\mathbf{u}) \mathbf{E}_{i}(\mathbf{u})=\sum_{i=1}^{3} \sum_{k=1}^{3} \frac{\partial T_{p}^{k}(\mathbf{u})}{\partial u^{j}} S_{k}^{i}(\tilde{\mathbf{x}}) \mathbf{E}_{i}(\mathbf{u}). \tag{5.7}$$

The sum over i in right hand side of the equality (5.7) can be calculated explicitly due to the first of the following two formulas:

$$\tilde{\mathbf{e}}_{k}=\sum_{i=1}^{3} S_{k}^{i} \mathbf{E}_{i}, \qquad\qquad \mathbf{E}_{p}=\sum_{k=1}^{3} T_{p}^{k} \tilde{\mathbf{e}}_{k}. \tag{5.8}$$

These formulas (5.8) relate the frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ and the basis vectors $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$ of the auxiliary Cartesian coordinate system. Now (5.7) is written as:

$$\sum_{i=1}^{3} \Gamma_{j p}^{i} \mathbf{E}_{i}=\sum_{k=1}^{3} \frac{\partial T_{p}^{k}(\mathbf{u})}{\partial u^{j}} \tilde{\mathbf{e}}_{k}=\sum_{k=1}^{3} \frac{\partial}{\partial u^{j}}\left(T_{p}^{k}(\mathbf{u}) \tilde{\mathbf{e}}_{k}\right).$$

The basis vector $\tilde{\mathbf{e}}_k$ does not depend on u^1, u^2, u^3. Therefore, it is brought into the brackets under the differentiation with respect to u^j. The sum over k in right hand side of the above formula is calculated explicitly due to the second formula (5.8). As a result the relationship (5.7) is transformed to the following one:

$$\frac{\partial \mathbf{E}_{p}}{\partial u^{j}}=\sum_{i=1}^{3} \Gamma_{j p}^{i} \cdot \mathbf{E}_{i}. \tag{5.9}$$

The formula (5.9) expresses the partial derivatives of the frame vectors back through these vectors. It can be understood as another one way for calculating the connection components Γ^i_{jp}. This formula comprises nothing related to the auxiliary Cartesian coordinates $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$. The vector $\mathbf{E}_p(u^1, u^2, u^3)$ is determined by the choice of curvilinear coordinates u^1, u^2, u^3 in the domain D. It is sufficient to differentiate this vector with respect to u^j and expand the resulting vector in the basis of the frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. Then the coefficients of this expansion yield the required values for Γ^i_{jp}. It is obvious that these values do not depend on the choice of the auxiliary Cartesian coordinates $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$ above. $\square$

Now let's proceed with deriving the rule for covariant differentiation of an arbitrary tensor field $\mathbf{A}$ of the type (r,s) in curvilinear coordinates. For this purpose we need another one expression for the connection components. It is derived from (5.5). Let's transform the formula (5.5) as follows:

$$\Gamma^i_{jp}(\mathbf{u})=\sum_{k=1}^{3}\frac{\partial}{\partial u^j}\left(S^i_k(\tilde{\mathbf{x}})\,T^k_p(\mathbf{u})\right)-\sum_{k=1}^{3}T^k_p(\mathbf{u})\,\frac{\partial S^i_k(\tilde{\mathbf{x}})}{\partial u^j}.$$

The matrices S and T are inverse to each other. Therefore, upon performing the summation over k in the first term we find that it vanishes. Hence, we get

$$\Gamma^i_{jp}(\mathbf{u})=-\sum_{k=1}^{3}T^k_p(\mathbf{u})\,\frac{\partial S^i_k(\tilde{\mathbf{x}})}{\partial u^j}. \tag{5.10}$$

Let $A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ be the components of a tensor field $\mathbf{A}$ of the type (r,s) in curvilinear coordinates. In order to calculate the components of $\mathbf{B}=\nabla\mathbf{A}$ we do the same maneuver as above. First of all we transform the components of $\mathbf{A}$ to some auxiliary Cartesian coordinate system:

$$\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}=\sum_{\substack{v_1\ldots\,v_r\\ w_1\ldots\,w_s}}T^{p_1}_{v_1}\ldots T^{p_r}_{v_r}\;S^{w_1}_{q_1}\ldots S^{w_s}_{q_s}\;A^{v_1\ldots\,v_r}_{w_1\ldots\,w_s}.$$

Then we calculate the components of the field $\mathbf{B}$ in this auxiliary Cartesian coordinate system simply by differentiating:

$$\tilde{B}^{p_1\ldots\,p_r}_{q_1\ldots\,q_{s+1}}=\sum_{\substack{v_1\ldots\,v_r\\ w_1\ldots\,w_s}}\frac{\partial\left(T^{p_1}_{v_1}\ldots T^{p_r}_{v_r}\;S^{w_1}_{q_1}\ldots S^{w_s}_{q_s}\;A^{v_1\ldots\,v_r}_{w_1\ldots\,w_s}\right)}{\partial\tilde{x}^{q_{s+1}}}.$$

Then we perform the inverse transformations of the components of $\mathbf{B}$ from the Cartesian coordinates back to the original curvilinear coordinate system:

$$\begin{aligned}B^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}&=\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_{s+1}}}S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\;T^{q_1}_{j_1}\ldots T^{q_{s+1}}_{j_{s+1}}\;\tilde{B}^{p_1\ldots\,p_r}_{q_1\ldots\,q_{s+1}}=\\&=\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_{s+1}}}\;\sum_{\substack{v_1\ldots\,v_r\\ w_1\ldots\,w_s}}S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\;T^{q_1}_{j_1}\ldots T^{q_{s+1}}_{j_{s+1}}\times\\&\qquad\times\frac{\partial\left(T^{p_1}_{v_1}\ldots T^{p_r}_{v_r}\;S^{w_1}_{q_1}\ldots S^{w_s}_{q_s}\;A^{v_1\ldots\,v_r}_{w_1\ldots\,w_s}\right)}{\partial\tilde{x}^{q_{s+1}}}.\end{aligned} \tag{5.11}$$

Applying the Leibniz rule for differentiating in (5.11), as a result we get three groups of summands. The summands of the first group correspond to differentiating the components of the matrix T, the summands of the second group arise when we differentiate the components of the matrix S in (5.11), and finally, the unique summand in the third group is produced by differentiating $A^{v_1\ldots\,v_r}_{w_1\ldots\,w_s}$. In any one of these summands if the term $T^{p_m}_{v_m}$ or the term $S^{w_n}_{q_n}$ is not differentiated, then this term is built into a sum that can be evaluated explicitly:

$$\sum_{p_m=1}^{3} S^{i_m}_{p_m}\,T^{p_m}_{v_m}=\delta^{i_m}_{v_m},\qquad\qquad \sum_{q_n=1}^{3} T^{q_n}_{j_n}\,S^{w_n}_{q_n}=\delta^{w_n}_{j_n}.$$

Therefore, one can evaluate explicitly the most part of the sums in the formula (5.11). Moreover, we have the following equality:

$$\sum_{q_{s+1}=1}^{3} T^{q_{s+1}}_{j_{s+1}}\,\frac{\partial}{\partial\tilde{x}^{q_{s+1}}}=\sum_{q_{s+1}=1}^{3}\frac{\partial\tilde{x}^{q_{s+1}}}{\partial u^{j_{s+1}}}\,\frac{\partial}{\partial\tilde{x}^{q_{s+1}}}=\frac{\partial}{\partial u^{j_{s+1}}}.$$

Taking into account all the above facts, we can bring (5.11) to

$$\begin{gathered}\nabla_{j_{s+1}}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{m=1}^{r}\sum_{v_m=1}^{3}\left(\sum_{p_m=1}^{3}S^{i_m}_{p_m}\,\frac{\partial T^{p_m}_{v_m}}{\partial u^{j_{s+1}}}\right)A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}+\\+\sum_{n=1}^{s}\sum_{w_n=1}^{3}\left(\sum_{q_n=1}^{3}T^{q_n}_{j_n}\,\frac{\partial S^{w_n}_{q_n}}{\partial u^{j_{s+1}}}\right)A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}+\frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial u^{j_{s+1}}}.\end{gathered}$$

Due to the formulas (5.5) and (5.10) one can express the sums enclosed into round brackets in the above equality through the Christoffel symbols. Ultimately, the formula (5.11) is brought to the following form:

$$\begin{gathered}\nabla_{j_{s+1}}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial u^{j_{s+1}}}+\\+\sum_{m=1}^{r}\sum_{v_m=1}^{3}\Gamma^{i_m}_{j_{s+1}\,v_m}\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}-\sum_{n=1}^{s}\sum_{w_n=1}^{3}\Gamma^{w_n}_{j_{s+1}\,j_n}\,A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}.\end{gathered}\tag{5.12}$$

The formula (5.12) is the rule for covariant differentiation of a tensorial field $\mathbf{A}$ of the type (r,s) in an arbitrary curvilinear coordinate system. This formula can be commented as follows: the covariant derivative $\nabla_{j_{s+1}}$ is obtained from the partial derivative $\partial/\partial u_{j_{s+1}}$ by adding $r+s$ terms — one per each index in the components of the field $\mathbf{A}$. The terms associated with the upper indices enter with the positive sign, the other terms associated with the lower indices enter with the negative sign. In such additional terms each of the upper indices i_m and each of the lower indices j_n are sequentially moved to the Christoffel symbol, while in its place we write the summation index v_m or w_n. The lower index j_{s+1} added as a result of covariant differentiation is always written as the first lower index in Christoffel symbols. The position of the summation indices v_m and w_n in Christoffel symbols is always complementary to their positions in the components of the field $\mathbf{A}$ so that they always form a pair of upper and lower indices. Though the formula (5.12) is rather huge, we hope that due to the above comments one can easily remember it and reproduce it in any particular case.

§ 6. Transformation of the connection components under a change of a coordinate system.

In deriving the formula for covariant differentiation of tensorial fields in curvilinear coordinates we discovered a new type of indexed objects — these are Christoffel symbols. The quantities Γ^k_{ij} are enumerated by one upper index and two lower indices, and their values are determined by the choice of a coordinate system. However, they are not the components of a tensorial fields of the type $(1,2)$. Indeed, the values of all Γ^k_{ij} in a Cartesian coordinate system are identically zero (this follows from the comparison of (5.12) with the formula (5.1) in Chapter II). But a tensorial field with purely zero components in some coordinate system cannot have nonzero components in any other coordinate system. Therefore, Christoffel symbols are the components of a non-tensorial geometric object which is called a *connection field* or simply a *connection*.

THEOREM 6.1. *Let u^1, u^2, u^3 and $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ be two coordinate systems in a domain $D \subset \mathbb{E}$. Then the connection components in these two coordinate systems are related to each other by means of the following equality:*

$$\Gamma^k_{ij} = \sum_{m=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} S^k_m\, T^p_i\, T^q_j\, \tilde{\Gamma}^m_{pq} + \sum_{m=1}^{3} S^k_m\, \frac{\partial T^m_i}{\partial u^j}. \tag{6.1}$$

Here S and T are the transition matrices given by the formulas (3.7).

A remark on the smoothness. The derivatives of the components of T in (6.1) and the formulas (3.7), where the components of T are defined as the partial derivatives of the transition functions (3.5), show that the connection components can be correctly defined only for coordinate systems of the smoothness class not lower than C^2. The same conclusion follows from the formula (5.5) for Γ^i_{jp}.

PROOF. In order to prove the theorem 6.1 we apply the formula (5.9). Let's write it for the frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, then apply the formula (3.6) for to express $\mathbf{E}_j$ through the vectors $\tilde{\mathbf{E}}_1$, $\tilde{\mathbf{E}}_2$, and $\tilde{\mathbf{E}}_3$:

$$\sum_{k=1}^{3} \Gamma^k_{ij}\, \mathbf{E}_k = \frac{\partial \tilde{\mathbf{E}}_j}{\partial u^i} = \sum_{m=1}^{3} \frac{\partial}{\partial u^i}\left(T^m_j\, \tilde{\mathbf{E}}_m\right). \tag{6.2}$$

Applying the Leibniz rule to the right hand side of (6.2), we get two terms:

$$\sum_{k=1}^{3} \Gamma^k_{ij}\, \mathbf{E}_k = \sum_{m=1}^{3} \frac{\partial T^m_j}{\partial u^i}\, \tilde{\mathbf{E}}_m + \sum_{q=1}^{3} T^q_j\, \frac{\partial \tilde{\mathbf{E}}_q}{\partial u^i}. \tag{6.3}$$

In the first term in the right hand side of (6.3) we express $\tilde{\mathbf{E}}_m$ through the vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$. In the second term we apply the chain rule and express the derivative with respect to u^i through the derivatives with respect to $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$:

$$\sum_{k=1}^{3} \Gamma^k_{ij}\, \mathbf{E}_k = \sum_{k=1}^{3}\sum_{m=1}^{3} S^k_m\, \frac{\partial T^m_j}{\partial u^i}\, \mathbf{E}_k + \sum_{q=1}^{3}\sum_{p=1}^{3} T^q_j\, \frac{\partial \tilde{u}^p}{\partial u^i}\, \frac{\partial \tilde{\mathbf{E}}_q}{\partial \tilde{u}^p}.$$

Now let's replace $\partial\tilde{u}^p/\partial u^i$ by T^p_i relying upon the formulas (3.7) and then apply the relationship (5.9) once more in the form of

$$\frac{\partial\tilde{\mathbf{E}}_q}{\partial\tilde{u}^p}=\sum_{m=1}^{3}\tilde{\Gamma}^m_{pq}\,\tilde{\mathbf{E}}_m.$$

As a result of the above transformations we can write the equality (6.3) as follows:

$$\sum_{k=1}^{3}\Gamma^k_{ij}\,\mathbf{E}_k=\sum_{k=1}^{3}\sum_{m=1}^{3}S^k_m\,\frac{\partial T^m_j}{\partial u^i}\,\mathbf{E}_k+\sum_{q=1}^{3}\sum_{p=1}^{3}\sum_{m=1}^{3}T^p_i\,T^q_j\,\tilde{\Gamma}^m_{pq}\,\tilde{\mathbf{E}}_m.$$

Now we need only to express $\tilde{\mathbf{E}}_m$ through the frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ and collect the similar terms in the above formula:

$$\sum_{k=1}^{3}\left(\Gamma^k_{ij}-\sum_{m=1}^{3}\sum_{q=1}^{3}\sum_{p=1}^{3}S^k_m\,T^p_i\,T^q_j\,\tilde{\Gamma}^m_{pq}-\sum_{m=1}^{3}S^k_m\,\frac{\partial T^m_j}{\partial u^i}\right)\mathbf{E}_k=0.$$

Since the frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are linearly independent, the expression enclosed into round brackets should vanish. As a result we get the equality exactly equivalent to the relationship (6.1) that we needed to prove. $\square$

§ 7. Concordance of metric and connection. Another formula for Christoffel symbols.

Let's consider the metric tensor $\mathbf{g}$. The covariant differential $\nabla\mathbf{g}$ of the field $\mathbf{g}$ is equal to zero (see formulas (6.7) in Chapter II). This is because in any Cartesian coordinates x^1, x^2, x^3 in $\mathbb{E}$ the components g_{ij} of the metric tensor do not depend on x^1, x^2, x^3. In a curvilinear coordinate system the components of the metric tensor $g_{ij}(u^1,u^2,u^3)$ usually are not constants. However, being equal to zero in Cartesian coordinates, the tensor $\nabla\mathbf{g}$ remains zero in any other coordinates:

$$\nabla_k g_{ij}=0. \tag{7.1}$$

The relationship (7.1) is known as the *concordance condition* for a metric and a connection. Taking into account (5.12), we can rewrite this condition as

$$\frac{g_{ij}}{\partial u^k}-\sum_{r=1}^{3}\Gamma^r_{ki}\,g_{rj}-\sum_{r=1}^{3}\Gamma^r_{kj}\,g_{ir}=0. \tag{7.2}$$

The formula (7.2) relates the connection components Γ^k_{ij} and the components of the metric tensor g_{ij}. Due to this relationship we can express Γ^k_{ij} through the components of the metric tensor provided we remember the following very important property of the connection components (5.5).

THEOREM 7.1. *The connection given by the formula* (5.5) *is a symmetric connection, i.e.* $\Gamma^k_{ij}=\Gamma^k_{j\,i}$.

PROOF. From (5.2) and (5.5) for Γ^k_{ij} we derive the following expression:

$$\Gamma^k_{ij}(\mathbf{u})=\sum_{q=1}^{3}S^k_q\,\frac{\partial T^q_j(\mathbf{u})}{\partial u^i}=\sum_{q=1}^{3}S^k_q\,\frac{\partial^2\tilde{x}^q}{\partial u^j\,\partial u^i}. \tag{7.3}$$

For the functions of the smoothness class C^2 the mixed second order partial derivatives do not depend on the order of differentiation:

$$\frac{\partial^2 \tilde{x}^q}{\partial u^j\,\partial u^i}=\frac{\partial^2 \tilde{x}^q}{\partial u^i\,\partial u^j}.$$

This fact immediately proves the symmetry of the Christoffel symbols given by the formula (7.3). Thus, the proof is over. $\square$

Now, returning back to the formula (7.2) relating Γ^k_{ij} and g_{ij}, we introduce the following notations that simplify the further calculations:

$$\Gamma_{ijk}=\sum_{r=1}^{3}\Gamma^r_{ij}\,g_{kr}. \tag{7.4}$$

It is clear that the quantities Γ_{ijk} in (7.4) are produced from Γ^k_{ij} by means of index lowering procedure described in Chapter II. Therefore, conversely, Γ^k_{ij} are obtained from Γ_{ijk} according to the following formula:

$$\Gamma^k_{ij}=\sum_{r=1}^{3}g^{kr}\,\Gamma_{ijr}. \tag{7.5}$$

From the symmetry of Γ^k_{ij} it follows that the quantities Γ_{ijk} in (7.4) are also symmetric with respect to the indices i and j, i.e. $\Gamma_{ijk}=\Gamma_{jik}$. Using the notations (7.4) and the symmetry of the metric tensor, the relationship (7.2) can be rewritten in the following way:

$$\frac{\partial g_{ij}}{\partial u^k}-\Gamma_{kij}-\Gamma_{kji}=0. \tag{7.6}$$

Let's complete (7.6) with two similar relationships applying two cyclic transpositions of the indices $i\to j\to k\to i$ to the formula (7.6). As a result we obtain

$$\begin{aligned}
&\frac{\partial g_{ij}}{\partial u^k}-\Gamma_{kij}-\Gamma_{kji}=0,\\
&\frac{\partial g_{jk}}{\partial u^i}-\Gamma_{ijk}-\Gamma_{ikj}=0,\\
&\frac{\partial g_{ki}}{\partial u^j}-\Gamma_{jki}-\Gamma_{jik}=0.
\end{aligned} \tag{7.7}$$

Let's add the last two relationships (7.7) and subtract the first one from the sum. Taking into account the symmetry of Γ_{ijk} with respect to i and j, we get

$$\frac{\partial g_{jk}}{\partial u^i}+\frac{\partial g_{ki}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k}-2\,\Gamma_{ijk}=0.$$

Using this equality, one can easily express Γ_{ijk} through the components of the metric tensor. Then one can substitute this expression into (7.5) and derive

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{r=1}^{3}g^{kr}\left(\frac{\partial g_{rj}}{\partial u^i}+\frac{\partial g_{ir}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^r}\right). \tag{7.8}$$

The relationship (7.8) is another formula for the Christoffel symbols Γ^k_{ij}, it follows from the symmetry of Γ^k_{ij} and from the concordance condition for the metric and connection. It is different from (5.5) and (5.10). The relationship (7.8) has the important advantage as compared to (5.5): one should not use an auxiliary Cartesian coordinate system for to apply it. As compared to (5.9), in (7.8) one should not deal with vector-functions $\mathbf{E}_i(u^1, u^2, u^3)$. All calculations in (7.8) are performed within a fixed curvilinear coordinate system provided the components of the metric tensor in this coordinate system are known.

§ 8. Parallel translation. The equation of a straight line in curvilinear coordinates.

Let $\mathbf{a}$ be a nonzero vector attached to some point A in the space $\mathbb{E}$. In a Euclidean space there is a procedure of parallel translation; applying this procedure one can bring the vector $\mathbf{a}$ from the point A to some other point B. This procedure does change neither the modulus nor the direction of the vector $\mathbf{a}$ being translated. In a Cartesian coordinate system the procedure of parallel translation is described in the most simple way: the original vector $\mathbf{a}$ at the point A and the translated vector $\mathbf{a}$ at the point B have the equal coordinates. In a curvilinear coordinate system the frame vectors at the point A and the frame vector at the point B form two different bases. Therefore, the components of the vector $\mathbf{a}$ in the following two expansions

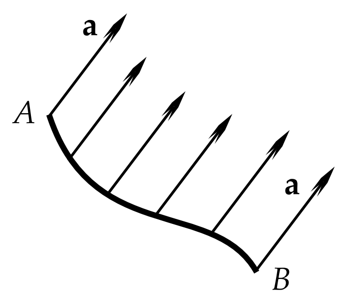

Fig. 8.1

$$\begin{aligned}\mathbf{a} &= a^1(A)\cdot\mathbf{E}_1(A) + a^2(A)\cdot\mathbf{E}_2(A) + a^3(A)\cdot\mathbf{E}_3(A),\\ \mathbf{a} &= a^1(B)\cdot\mathbf{E}_1(B) + a^2(B)\cdot\mathbf{E}_2(B) + a^3(B)\cdot\mathbf{E}_3(B)\end{aligned}\tag{8.1}$$

in general case are different. If the points A and B are closed to each other, then the triples of vectors $\mathbf{E}_1(A)$, $\mathbf{E}_2(A)$, $\mathbf{E}_3(A)$ and $\mathbf{E}_1(B)$, $\mathbf{E}_2(B)$, $\mathbf{E}_3(B)$ are approximately the same. Hence, in this case the components of the vector $\mathbf{a}$ in the expansions (8.1) are slightly different from each other. This consideration shows that in curvilinear coordinates the parallel translation should be performed gradually: one should first converge the point B with the point A, then slowly move the point B toward its ultimate position and record the coordinates of the vector $\mathbf{a}$ in the second expansion (8.1) at each intermediate position of the point B. The most simple way to implement this plan is to link A and B with some smooth parametric curve $\mathbf{r} = \mathbf{r}(t)$, where $t \in [0,\,1]$. In a curvilinear coordinate system a parametric curve is given by three functions $u^1(t)$, $u^2(t)$, $u^3(t)$ that for each $t \in [0,\,1]$ yield the coordinates of the corresponding point on the curve.

THEOREM 8.1. *For a parametric curve given by three functions $u^1(t)$, $u^2(t)$, and $u^3(t)$ in some curvilinear coordinate system the components of the tangent vector $\boldsymbol{\tau}(t)$ in the moving frame of that coordinate system are determined by the derivatives $\dot u^1(t)$, $\dot u^2(t)$, $\dot u^3(t)$.*

Proof. Curvilinear coordinates u^1, u^2, u^3 determine the position of a point in the space by means of the vector-function $\mathbf{r} = \mathbf{r}(u^1, u^2, u^3)$, where $\mathbf{r}$ is the radius-vector of that point in some auxiliary Cartesian coordinate system (see formulas (2.4) and (2.5)). Therefore, the vectorial-parametric equation of the curve is represented in the following way:

$$\mathbf{r} = \mathbf{r}(u^1(t), u^2(t), u^3(t)). \tag{8.2}$$

Applying the chain rule to the function $\mathbf{r}(t)$ in (8.2), we get

$$\boldsymbol{\tau}(t) = \frac{d\mathbf{r}}{dt} = \sum_{j=1}^{3} \frac{\partial \mathbf{r}}{\partial u^j} \cdot \dot{u}^j(t). \tag{8.3}$$

Remember that due to the formula (2.7) the partial derivatives in (8.3) coincide with the frame vectors of the curvilinear coordinate system. Therefore the formula (8.3) itself can be rewritten as follows:

$$\boldsymbol{\tau}(t) = \sum_{j=1}^{3} \dot{u}^j(t) \cdot \mathbf{E}_j(u^1(t), u^2(t), u^3(t)). \tag{8.4}$$

It is easy to see that (8.4) is the expansion of the tangent vector $\boldsymbol{\tau}(t)$ in the basis formed by the frame vectors of the curvilinear coordinate system. The components of the vector $\boldsymbol{\tau}(t)$ in the expansion (8.4) are the derivatives $\dot{u}^1(t)$, $\dot{u}^2(t)$, $\dot{u}^3(t)$. The theorem is proved. $\square$

Let's apply the procedure of parallel translation to the vector $\mathbf{a}$ and translate this vector to all points of the curve linking the points A and B (see Fig. 8.1). Then we can write the following expansion for this vector

$$\mathbf{a} = \sum_{i=1}^{3} a^i(t) \cdot \mathbf{E}_i(u^1(t), u^2(t), u^3(t)). \tag{8.5}$$

This expansion is analogous to (8.4). Let's differentiate the relationship (8.5) with respect to the parameter t and take into account that $\mathbf{a} = \text{const}$:

$$0 = \frac{d\mathbf{a}}{dt} = \sum_{i=1}^{3} \dot{a}^i \cdot \mathbf{E}_i + \sum_{i=1}^{3} \sum_{j=1}^{3} a^i \, \frac{\partial \mathbf{E}_i}{\partial u^j} \, \dot{u}^j.$$

Now let's use the formula (5.9) in order to differentiate the frame vectors of the curvilinear coordinate system. As a result we derive

$$\sum_{i=1}^{3} \left(\dot{a}^i + \sum_{j=1}^{3} \sum_{k=1}^{3} \Gamma^i_{jk} \, \dot{u}^j \, a^k \right) \cdot \mathbf{E}_i = 0.$$

Since the frame vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are linearly independent, we obtain

$$\dot{a}^i + \sum_{j=1}^{3} \sum_{k=1}^{3} \Gamma^i_{jk} \, \dot{u}^j \, a^k = 0. \tag{8.6}$$

The equation (8.6) is called the *differential equation of the parallel translation of a vector along a curve*. This is the system of three linear differential equations of the first order with respect to the components of the vector $\mathbf{a}$. Actually, in order to perform the parallel translation of a vector $\mathbf{a}$ from the point A to the point B in curvilinear coordinates one should set the initial data for the components of the vector $\mathbf{a}$ at the point A (i. e. for $t = 0$) and then solve the Cauchy problem for the equations (8.6).

The procedure of the parallel translation of vectors along curves leads us to the situation where at each point of a curve in $\mathbb{E}$ we have some vector attached to that point. The same situation arises in considering the vectors $\boldsymbol{\tau}$, $\mathbf{n}$, and $\mathbf{b}$ that form the Frenet frame of a curve in $\mathbb{E}$ (see Chapter I). Generalizing this situation one can consider the set of tensors of the type (r, s) attached to the points of some curve. Defining such a set of tensors differs from defining a tensorial field in $\mathbb{E}$ since in order to define a tensor field in $\mathbb{E}$ one should attach a tensor to each point of the space, not only to the points of a curve. In the case, where the tensors of the type (r, s) are defined only at the points of a curve, we say that a *tensor field of the type* (r, s) *on a curve* is given. In order to write the components of such a tensor field $\mathbf{A}$ we can use the moving frame $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ of some curvilinear coordinate system in some neighborhood of the curve. These components form a set of functions of the scalar parameter t specifying the points of the curve:

$$A^{i_1\dots\, i_r}_{j_1\dots\, j_s} = A^{i_1\dots\, i_r}_{j_1\dots\, j_s}(t). \tag{8.7}$$

Under a change of curvilinear coordinate system the quantities (8.7) are transformed according to the standard rule

$$\begin{aligned} A^{i_1\dots\, i_r}_{j_1\dots\, j_s}(t) = \sum_{\substack{p_1\dots\, p_r\\ q_1\dots\, q_s}} &S^{i_1}_{p_1}(t)\dots\, S^{i_r}_{p_r}(t)\times\\ &\times T^{q_1}_{j_1}(t)\dots\, T^{q_s}_{j_s}(t)\ \tilde{A}^{p_1\dots\, p_r}_{q_1\dots\, q_s}(t), \end{aligned}\tag{8.8}$$

where $S(t)$ and $T(t)$ are the values of the transition matrices at the points of the curve. They are given by the following formulas:

$$\begin{aligned} &S(t) = S(\tilde{u}^1(t), \tilde{u}^2(t), \tilde{u}^3(t)),\\ &T(t) = T(u^1(t), u^2(t), u^3(t)). \end{aligned}\tag{8.9}$$

We cannot use the formula (5.12) for differentiating the field $\mathbf{A}$ on the curve since the only argument, which the functions (8.7) depend on, is the parameter t. Therefore, we need to modify the formula (5.12) as follows:

$$\begin{aligned} \nabla_t A^{i_1\dots\, i_r}_{j_1\dots\, j_s} = \frac{dA^{i_1\dots\, i_r}_{j_1\dots\, j_s}}{dt} &+\\ + \sum^{r}_{m=1}\sum^{3}_{q=1}\sum^{3}_{v_m=1} \Gamma^{i_m}_{q\,v_m}\ \dot{u}^q\ A^{i_1\dots\, v_m\dots\, i_r}_{j_1\dots\, j_s} &- \sum^{s}_{n=1}\sum^{3}_{q=1}\sum^{3}_{w_n=1} \Gamma^{w_n}_{q\,j_n}\ \dot{u}^q\ A^{i_1\dots\, i_r}_{j_1\dots\, w_n\dots\, j_s}. \end{aligned}\tag{8.10}$$

The formula (8.10) expresses the *rule for covariant differentiation of a tensor field* $\mathbf{A}$ *with respect to the parameter* t *along a parametric curve* in curvilinear coordinates u^1, u^2, u^3. Unlike (5.12), the index t beside the nabla sign is not an additional index. It is set only for to denote the variable t with respect to which the differentiation in the formula (8.10) is performed.

THEOREM 8.2. *Under a change of coordinates u^1, u^2, u^3 for other coordinates $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ the quantities $B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \nabla_t A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ calculated by means of the formula (8.10) are transformed according to the rule (8.8) and define a tensor field $\mathbf{B} = \nabla_t\mathbf{A}$ of the type (r, s) which is called the covariant derivative of the field $\mathbf{A}$ with respect to the parameter t along a curve.*

PROOF. The proof of this theorem is pure calculations. Let's begin with the first term in (8.10). Let's express $A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ through the components of the field $\mathbf{A}$ in the other coordinates $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{u}^3$ by means of (8.8). In calculating $dA^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}/dt$ this is equivalent to differentiating both sides of (8.8) with respect to t:

$$
\begin{aligned}
\frac{dA^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{dt} &= \sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}} S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\,\ldots\,T^{q_s}_{j_s}\,\frac{d\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}}{dt}+\\
&+\sum_{m=1}^{r}\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}} S^{i_1}_{p_1}\ldots\,\dot{S}^{i_m}_{p_m}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\,\ldots\,T^{q_s}_{j_s}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}+\\
&+\sum_{n=1}^{s}\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}} S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\,\ldots\,\dot{T}^{q_n}_{j_n}\,\ldots\,T^{q_s}_{j_s}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}.
\end{aligned}
\tag{8.11}
$$

For to calculate the derivatives $\dot{S}^{i_m}_{p_m}$ and $\dot{T}^{q_n}_{j_n}$ in (8.11) we use the fact that the transition matrices S and T are inverse to each other:

$$
\begin{aligned}
\dot{S}^{i_m}_{p_m} &= \sum_{k=1}^{3}\sum_{v_m=1}^{3}\dot{S}^{i_m}_{k}\,T^{k}_{v_m}\,S^{v_m}_{p_m} = \sum_{v_m=1}^{3}\sum_{k=1}^{3}\frac{d\left(S^{i_m}_{k}\,T^{k}_{v_m}\right)}{dt}\,S^{v_m}_{p_m}-\\
&-\sum_{v_m=1}^{3}\sum_{k=1}^{3}S^{i_m}_{k}\,\frac{dT^{k}_{v_m}}{dt}\,S^{v_m}_{p_m} = -\sum_{v_m=1}^{3}\left(\sum_{k=1}^{3}S^{i_m}_{k}\,\frac{dT^{k}_{v_m}}{dt}\right)S^{v_m}_{p_m},\\
\dot{T}^{q_n}_{j_n} &= \sum_{k=1}^{3}\sum_{w_n=1}^{3}\dot{T}^{k}_{j_n}\,S^{w_n}_{k}\,T^{q_n}_{w_n} = \sum_{w_n=1}^{3}\left(\sum_{k=1}^{3}\frac{dT^{k}_{j_n}}{dt}\,S^{w_n}_{k}\right)T^{q_n}_{w_n}.
\end{aligned}
$$

In order to transform further the above formulas for the derivatives $\dot{S}^{i_m}_{p_m}$ and $\dot{T}^{q_n}_{j_n}$ we use the second formula in (8.9):

$$
\dot{S}^{i_m}_{p_m} = -\sum_{v_m=1}^{3}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}S^{i_m}_{k}\,\frac{\partial T^{k}_{v_m}}{\partial u^q}\,\dot{u}^q\right)S^{v_m}_{p_m},
\tag{8.12}
$$

$$
\dot{T}^{q_n}_{j_n} = \sum_{w_n=1}^{3}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}S^{w_n}_{k}\,\frac{\partial T^{k}_{j_n}}{\partial u^q}\,\dot{u}^q\right)T^{q_n}_{w_n}.
\tag{8.13}
$$

Let's substitute (8.12) and (8.13) into (8.11). Then, taking into account the relationship (8.8), we can perform the summation over $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$

in the second and the third terms in (8.11) thus transforming (8.11) to

$$\begin{aligned}\frac{dA^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{dt}&=\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}}S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\,\frac{d\tilde A^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}}{dt}-\\&-\sum^{r}_{m=1}\sum^{3}_{q=1}\sum^{3}_{v_m=1}\left(\sum^{3}_{k=1}S^{i_m}_k\,\frac{\partial T^k_{v_m}}{\partial u^q}\right)\dot u^q\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}+\\&+\sum^{s}_{n=1}\sum^{3}_{q=1}\sum^{3}_{w_n=1}\left(\sum^{3}_{k=1}S^{w_n}_k\,\frac{\partial T^k_{j_n}}{\partial u^q}\right)\dot u^q\,A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}.\end{aligned}\tag{8.14}$$

The second and the third terms in (8.10) and (8.14) are similar in their structure. Therefore, one can collect the similar terms upon substituting (8.14) into (8.10). Collecting these similar terms, we get the following two expressions

$$\Gamma^{i_m}_{q\,v_m}-\sum^{3}_{k=1}S^{i_m}_k\,\frac{\partial T^k_{v_m}}{\partial u^q},\qquad\qquad \Gamma^{w_n}_{q\,j_n}-\sum^{3}_{k=1}S^{w_n}_k\,\frac{\partial T^k_{j_n}}{\partial u^q}$$

as the coefficients. Let's apply (6.1) to these expressions:

$$\begin{aligned}\Gamma^{i_m}_{q\,v_m}-\sum^{3}_{k=1}S^{i_m}_k\,\frac{\partial T^k_{v_m}}{\partial u^q}&=\sum^{3}_{r=1}\sum^{3}_{p=1}\sum^{3}_{k=1}S^{i_m}_r\,\tilde\Gamma^r_{pk}\,T^p_q\,T^k_{v_m},\\ \Gamma^{w_n}_{q\,j_n}-\sum^{3}_{k=1}S^{w_n}_k\,\frac{\partial T^k_{j_n}}{\partial u^q}&=\sum^{3}_{r=1}\sum^{3}_{p=1}\sum^{3}_{k=1}S^{w_n}_r\,\tilde\Gamma^r_{pk}\,T^p_q\,T^k_{j_n}.\end{aligned}\tag{8.15}$$

If we take into account (8.15) when substituting (8.14) into (8.10), then the equality (8.10) is written in the following form:

$$\begin{aligned}\nabla_t A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}}S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\,\frac{d\tilde A^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}}{dt}+\\&+\sum^{r}_{m=1}\sum^{3}_{q=1}\sum^{3}_{v_m=1}\sum^{3}_{p_m=1}\sum^{3}_{p=1}\sum^{3}_{k=1}S^{i_m}_{p_m}\,\tilde\Gamma^{p_m}_{pk}\,T^p_q\,T^k_{v_m}\,\dot u^q\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}-\\&-\sum^{s}_{n=1}\sum^{3}_{q=1}\sum^{3}_{w_n=1}\sum^{3}_{q_n=1}\sum^{3}_{p=1}\sum^{3}_{k=1}S^{w_n}_k\,\tilde\Gamma^k_{p\,q_n}\,T^p_q\,T^{q_n}_{j_n}\,\dot u^q\,A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}.\end{aligned}$$

In order to transform it further we express $A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}$ and $A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}$ through the components of the field $\mathbf{A}$ in the other coordinate system by means of the formula (8.8). Moreover, we take into account that $T^p_q\,\dot u^q$ upon summing up over q yields $\dot{\tilde u}^p$. As a result we obtain:

$$\begin{aligned}\nabla_t A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}}S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\Biggl(\frac{d\tilde A^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}}{dt}+\\+\sum^{r}_{m=1}\sum^{3}_{p=1}\sum^{3}_{v_m=1}\tilde\Gamma^{p_m}_{p\,v_m}\,\dot{\tilde u}^p\,\tilde A^{p_1\ldots\,v_m\ldots\,p_r}_{q_1\ldots\,q_s}-\sum^{s}_{n=1}\sum^{3}_{p=1}\sum^{3}_{w_n=1}\tilde\Gamma^{w_n}_{p\,q_n}\,\dot{\tilde u}^p\,\tilde A^{p_1\ldots\,p_r}_{q_1\ldots\,w_n\ldots\,q_s}\Biggr).\end{aligned}\tag{8.16}$$

Note that the expression enclosed into round brackets in (8.16) is $\nabla_t \tilde{A}^{p_1 \dots\, p_r}_{q_1 \dots\, q_s}$ exactly. Therefore, the formula (8.16) means that the components of the field $\nabla_t \mathbf{A}$ on a curve calculated according to the formula (8.10) obey the transformation rule (8.8). Thus, the theorem 8.2 is proved. $\square$

Now let's return to the formula (8.6). The left hand side of this formula coincides with the expression (8.10) for the covariant derivative of the vector field $\mathbf{a}$ with respect to the parameter t. Therefore, the equation of parallel translation can be written as $\nabla_t \mathbf{a} = 0$. In this form, the equation of parallel translation can be easily generalized for the case of an arbitrary tensor $\mathbf{A}$:

$$\nabla_t \mathbf{A} = 0. \tag{8.17}$$

The equation (8.17) cannot be derived directly since the procedure of parallel translation for arbitrary tensors has no visual representation like Fig. 8.1.

Let's consider a segment of a straight line given parametrically by the functions $u^1(t)$, $u^2(t)$, $u^3(t)$ in a curvilinear coordinates. Let $t = s$ be the natural parameter on this straight line. Then the tangent vector $\boldsymbol{\tau}(t)$ is a vector of the unit length at all points of the line. Its direction is also unchanged. Therefore, its components $\dot{u}^i$ satisfy the equation of parallel translation. Substituting $a^i = \dot{u}^i$ into (8.6), we get

$$\ddot{u}^i + \sum_{j=1}^{3} \sum_{k=1}^{3} \Gamma^i_{jk}\, \dot{u}^j\, \dot{u}^k = 0. \tag{8.18}$$

The equation (8.18) is the differential equation of a straight line in curvilinear coordinates (written for the natural parametrization $t = s$).

§ 9. Some calculations in polar, cylindrical, and spherical coordinates.

Let's consider the polar coordinate system on a plane. It is given by formulas (1.1). Differentiating the expressions (1.1), we find the components of the frame vectors for the polar coordinate system:

$$\mathbf{E}_1 = \begin{Vmatrix} \cos(\varphi) \\ \sin(\varphi) \end{Vmatrix}, \qquad \mathbf{E}_2 = \begin{Vmatrix} -\rho \sin(\varphi) \\ \rho \cos(\varphi) \end{Vmatrix}. \tag{9.1}$$

The column-vectors (9.1) are composed by the coordinates of the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ in the orthonormal basis. Therefore, we can calculate their scalar products and thus find the components of direct and inverse metric tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$:

$$g_{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & \rho^2 \end{Vmatrix}, \qquad g^{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & \rho^{-2} \end{Vmatrix}. \tag{9.2}$$

Once the components of $\mathbf{g}$ and $\hat{\mathbf{g}}$ are known, we can calculate the Christoffel symbols. For this purpose we apply the formula (7.8):

$$\begin{aligned} &\Gamma^1_{11} = 0, && \Gamma^1_{12} = \Gamma^1_{21} = 0, && \Gamma^1_{22} = -\rho, \\ &\Gamma^2_{11} = 0, && \Gamma^2_{12} = \Gamma^2_{21} = \rho^{-1}, && \Gamma^2_{22} = 0. \end{aligned} \tag{9.3}$$

Let's apply the connection components (9.3) in order to calculate the Laplace operator $\triangle$ in polar coordinates. Let ψ be some scalar field: $\psi = \psi(\rho, \varphi)$. Then

$$\triangle\psi = \sum_{i=1}^{2}\sum_{j=1}^{2} g^{ij}\left(\frac{\partial^2\psi}{\partial u^i\,\partial u^j} - \sum_{k=1}^{2}\Gamma^k_{ij}\,\frac{\partial\psi}{\partial u^k}\right). \tag{9.4}$$

The formula (9.4) is a two-dimensional version of the formula (10.15) from Chapter II applied to a scalar field. Substituting (9.3) into (9.4), we get

$$\triangle\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2\psi}{\partial\varphi^2}. \tag{9.5}$$

Now let's consider the cylindrical coordinate system. For the components of metric tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$ in this case we have

$$g_{ij} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \qquad\qquad g^{ij} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{9.6}$$

From (9.6) by means of (7.8) we derive the connection components:

$$\begin{aligned} &\Gamma^1_{11} = 0, && \Gamma^1_{12} = 0, && \Gamma^1_{21} = 0,\\ &\Gamma^1_{13} = 0, && \Gamma^1_{31} = 0, && \Gamma^1_{22} = -\rho,\\ &\Gamma^1_{23} = 0, && \Gamma^1_{32} = 0, && \Gamma^1_{33} = 0, \end{aligned} \tag{9.7}$$

$$\begin{aligned} &\Gamma^2_{11} = 0, && \Gamma^2_{12} = \rho^{-1}, && \Gamma^2_{21} = \rho^{-1},\\ &\Gamma^2_{13} = 0, && \Gamma^2_{31} = 0, && \Gamma^2_{22} = 0,\\ &\Gamma^2_{23} = 0, && \Gamma^2_{32} = 0, && \Gamma^2_{33} = 0, \end{aligned} \tag{9.8}$$

$$\begin{aligned} &\Gamma^3_{11} = 0, && \Gamma^3_{12} = 0, && \Gamma^3_{21} = 0,\\ &\Gamma^3_{13} = 0, && \Gamma^3_{31} = 0, && \Gamma^3_{22} = 0,\\ &\Gamma^3_{23} = 0, && \Gamma^3_{32} = 0, && \Gamma^3_{33} = 0. \end{aligned} \tag{9.9}$$

Let's rewrite in the dimension 3 the relationship (9.4) for the Laplace operator applied to a scalar field ψ:

$$\triangle\psi = \sum_{i=1}^{3}\sum_{j=1}^{3} g^{ij}\left(\frac{\partial^2\psi}{\partial u^i\,\partial u^j} - \sum_{k=1}^{3}\Gamma^k_{ij}\,\frac{\partial\psi}{\partial u^k}\right). \tag{9.10}$$

Substituting (9.7), (9.8), and (9.9) into the formula (9.10), we get

$$\triangle\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial h^2}. \tag{9.11}$$

Now we derive the formula for the components of rotor in cylindrical coordinates. Let $\mathbf{A}$ be a vector field and let A^1, A^2, A^3 be its components in cylindrical coordinates. In order to calculate the components of the field $\mathbf{F} = \operatorname{rot} \mathbf{A}$ we use the formula (10.5) from Chapter II. This formula comprises the volume tensor whose components are calculated by formula (8.1) from Chapter II. The sign factor ξ_E in this formula is determined by the orientation of a coordinate system. The cylindrical coordinate system can be either right-oriented or left-oriented. It depends on the orientation of the auxiliary Cartesian coordinate system x^1, x^2, x^3 which is related to the cylindrical coordinates by means of the relationships (1.3). For the sake of certainty we assume that the right-oriented cylindrical coordinates are chosen. Then $\xi_E = 1$ and for the components of the rotor $\mathbf{F} = \operatorname{rot} \mathbf{A}$ we derive

$$F^m = \sqrt{\det \mathbf{g}}\,\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{mi}\,\varepsilon_{ijk}\,g^{jq}\,\nabla_q F^k. \tag{9.12}$$

Taking into account (9.7), (9.8), (9.9), (9.6) and using (9.12), we get

$$\begin{aligned} F^1 &= \frac{1}{\rho}\,\frac{\partial A^3}{\partial \varphi} - \rho\,\frac{\partial A^2}{\partial h},\\ F^2 &= \frac{1}{\rho}\,\frac{\partial A^1}{\partial h} - \frac{1}{\rho}\,\frac{\partial A^3}{\partial \rho},\\ F^3 &= \rho\,\frac{\partial A^2}{\partial \rho} - \frac{1}{\rho}\,\frac{\partial A^1}{\partial \varphi} + 2\,A^2. \end{aligned} \tag{9.13}$$

The relationships (9.13) can be written in form of the determinant:

$$\operatorname{rot} \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial h}\\ A^1 & \rho^2 A^2 & A^3 \end{vmatrix}, \tag{9.14}$$

Here $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are the frame vectors of the cylindrical coordinates.

In the case of spherical coordinates, we begin the calculations by deriving the formula for the components of the metric tensor $\mathbf{g}$:

$$g_{ij} = \begin{Vmatrix} 1 & 0 & 0\\ 0 & \rho^2 & 0\\ 0 & 0 & \rho^2\sin^2(\vartheta) \end{Vmatrix}. \tag{9.15}$$

Then we calculate the connection components and write then in form of the array:

$$\begin{aligned} &\Gamma^1_{11} = 0, && \Gamma^1_{12} = 0, && \Gamma^1_{21} = 0,\\ &\Gamma^1_{13} = 0, && \Gamma^1_{31} = 0, && \Gamma^1_{22} = -\rho,\\ &\Gamma^1_{23} = 0, && \Gamma^1_{32} = 0, && \Gamma^1_{33} = -\rho\sin^2(\vartheta), \end{aligned} \tag{9.16}$$

$$\begin{aligned}
&\Gamma^2_{11} = 0, &&\Gamma^2_{12} = \rho^{-1}, &&\Gamma^2_{21} = \rho^{-1},\\
&\Gamma^2_{13} = 0, &&\Gamma^2_{31} = 0, &&\Gamma^2_{22} = 0,\\
&\Gamma^2_{23} = 0, &&\Gamma^2_{32} = 0, &&\Gamma^2_{33} = -\frac{\sin(2\vartheta)}{2},
\end{aligned}\tag{9.17}$$

$$\begin{aligned}
&\Gamma^3_{11} = 0, &&\Gamma^3_{12} = 0, &&\Gamma^3_{21} = 0,\\
&\Gamma^3_{13} = \rho^{-1}, &&\Gamma^3_{31} = \rho^{-1}, &&\Gamma^3_{22} = 0,\\
&\Gamma^3_{23} = \cot(\vartheta), &&\Gamma^3_{32} = \cot(\vartheta), &&\Gamma^3_{33} = 0.
\end{aligned}\tag{9.18}$$

Substituting (9.16), (9.17), and (9.18) into the relationship (9.10), we get

$$\triangle\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{2}{\rho}\,\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2\psi}{\partial\vartheta^2} + \frac{\cot(\vartheta)}{\rho^2}\,\frac{\partial\psi}{\partial\vartheta} + \frac{1}{\rho^2\sin^2(\vartheta)}\,\frac{\partial^2\psi}{\partial\varphi^2}.\tag{9.19}$$

Let $\mathbf{A}$ be a vector field with the components A^1, A^2, A^3 in the right-oriented spherical coordinates. Denote $\mathbf{F} = \operatorname{rot}\mathbf{A}$. Then for the components of $\mathbf{F}$ we get

$$\begin{aligned}
F^1 &= \sin(\vartheta)\,\frac{\partial A^3}{\partial\vartheta} - \frac{1}{\sin(\vartheta)}\,\frac{\partial A^2}{\partial\varphi} + 2\,\cos(\vartheta)\,A^3,\\
F^2 &= \frac{1}{\rho^2\sin(\vartheta)}\,\frac{\partial A^1}{\partial\varphi} - \sin(\vartheta)\,\frac{\partial A^3}{\partial\rho} - \frac{2\sin(\vartheta)}{\rho}\,A^3,\\
F^3 &= \frac{1}{\sin(\vartheta)}\,\frac{\partial A^2}{\partial\rho} - \frac{1}{\rho^2\sin(\vartheta)}\,\frac{\partial A^1}{\partial\vartheta} + \frac{2}{\rho\sin(\vartheta)}\,A^2.
\end{aligned}\tag{9.20}$$

Like (9.13), the formulas (9.20) can be written in form of the determinant:

$$\operatorname{rot}\mathbf{A} = \frac{\rho^{-2}}{\sin(\vartheta)}\begin{vmatrix}
\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\\[1ex]
\dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\vartheta} & \dfrac{\partial}{\partial\varphi}\\[2ex]
A^1 & \rho^2 A^2 & \rho^2\sin^2(\vartheta)A^3
\end{vmatrix}.\tag{9.21}$$

The formulas (9.5), (9.11), and (9.19) for the Laplace operator and the formulas (9.14) and (9.21) for the rotor is the main goal of the calculations performed just above in this section. They are often used in applications and can be found in some reference books for engineering computations.

The matrices $\mathbf{g}$ in all of the above coordinate systems are diagonal. Such coordinate systems are called *orthogonal*, while the quantities $H_i = \sqrt{g_{ii}}$ are called the *Lame coefficients* of orthogonal coordinates. Note that there is no orthonormal curvilinear coordinate system. All such systems are necessarily Cartesian, this fact follows from (7.8) and (5.9).

CHAPTER IV

GEOMETRY OF SURFACES.

§ 1. Parametric surfaces. Curvilinear coordinates on a surface.

A surface is a two-dimensional spatially extended geometric object. There are several ways for expressing quantitatively (mathematically) this fact of two-dimensionality of surfaces. In the three-dimensional Euclidean space $\mathbb{E}$ the choice of an arbitrary point implies three degrees of freedom: a point is determined by three coordinates. In order to decrease this extent of arbitrariness we can bind three coordinates of a point by an equation:

$$F(x^1, x^2, x^3) = 0. \tag{1.1}$$

Then the choice of two coordinates determines the third coordinate of a point. This means that we can define a surface by means of an equation in some coordinate system (for the sake of simplicity we can choose a Cartesian coordinate system). We have already used this method of defining surfaces (see formula (1.2) in Chapter I) when we defined a curve as an intersection of two surfaces.

Another way of defining a surface is the *parametric method*. Unlike curves, surfaces are parameterized by two parameters. Let's denote them u^1 and u^2:

$$\mathbf{r} = \mathbf{r}(u^1, u^2) = \begin{Vmatrix} x^1(u^1, u^2) \\ \\ x^2(u^1, u^2) \\ \\ x^3(u^1, u^2) \end{Vmatrix}. \tag{1.2}$$

The formula (1.2) expresses the radius-vector of the points of a surface in some Cartesian coordinate system as a function of two parameters u^1, u^2. Usually, only a part of a surface is represented in parametric form. Therefore, considering the pair of numbers (u^1, u^2) as a point of $\mathbb{R}^2$, we can assume that the point (u^1, u^2) runs over some domain $U \subset \mathbb{R}^2$. Let's denote by D the image of the domain U under the mapping (1.2). Then D is the *domain being mapped*, U is the *map* or the *chart*, and (1.2) is the *chart mapping*: it maps U onto D.

The smoothness class of the surface D is determined by the smoothness class of the functions $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, and $x^3(u^1, u^2)$ in formula (1.2). In what fallows we shall consider only those surfaces for which these functions are at least continuously differentiable. Then, differentiating these functions, we can arrange their derivatives into the Jacobi matrix:

$$I = \begin{Vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} \end{Vmatrix}. \tag{1.3}$$

Definition 1.1. A continuously differentiable mapping (1.2) is called *regular* at a point (u^1, u^2) if the rank of the Jacobi matrix (1.3) calculated at that point is equal to 2.

Definition 1.2. A set D is called a *regular fragment* of a continuously differentiable surface if there is a mapping $\mathbf{u}\colon D \to U$ from D to some domain $U \subset \mathbb{R}^2$ and the following conditions are fulfilled:

(1) the mapping $\mathbf{u} : D \to U$ is bijective;
(2) the inverse mapping $\mathbf{u}^{-1}\colon U \to D$ given by three continuously differentiable functions (1.2) is regular at all points of the domain U.

The Jacobi matrix (1.3) has three minors of the order 2. These are the determinants of the following 2×2 matrices:

$$\begin{vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} \end{vmatrix}, \qquad \begin{vmatrix} \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} \end{vmatrix}, \qquad \begin{vmatrix} \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} \\ \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} \end{vmatrix}. \tag{1.4}$$

In the case of regularity of the mapping (1.2) at least one of the determinants (1.4) is nonzero. At the expense of renaming the variables x^1, x^2, x^3 we always can do so that the first determinant will be nonzero:

$$\begin{vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} \end{vmatrix} \neq 0. \tag{1.5}$$

In this case we consider the first two functions $x^1(u^2, u^2)$ and $x^2(u^2, u^2)$ in (1.2) as a mapping and write them as follows:

$$\begin{cases} x^1 = x^1(u^1, u^2), \\ x^2 = x^2(u^1, u^2). \end{cases} \tag{1.6}$$

Due to (1.5) the mapping (1.6) is locally invertible. Upon restricting (1.6) to some sufficiently small neighborhood of an arbitrary preliminarily chosen point one can construct two continuously differentiable functions

$$\begin{cases} u^1 = u^1(x^1, x^2), \\ u^2 = u^2(x^1, x^2) \end{cases} \tag{1.7}$$

that implement the inverse mapping for (1.6). This fact is well-known, it is a version of the theorem on implicit functions (see [2], see also the theorem 2.1 in Chapter III). Let's substitute u^1 and u^2 from (1.7) into the arguments of the third function $x^3(u^1, u^2)$ in the formula (1.2). As a result we obtain the function $F(x^1, x^2) = x^3(u^1(x^2, x^2), u^2(x^2, x^2))$ such that each regular fragment of a surface can locally (i. e. in some neighborhood of each its point) be presented as a graph of a continuously differentiable function of two variables:

$$x^3 = F(x^1, x^2). \tag{1.8}$$

A remark on singular points. If we give up the regularity condition from the definition 1.2, this may cause the appearance of singular points on a surface. As an example we consider two surfaces given by smooth functions:

$$\begin{cases} x^1 = (u^1)^3, \\ x^2 = (u^2)^3, \\ x^3 = (u^1)^2 + (u^2)^2, \end{cases} \qquad \begin{cases} x^1 = (u^1)^3, \\ x^2 = (u^2)^3, \\ x^3 = (u^1)^4 + (u^2)^4. \end{cases} \tag{1.9}$$

In both cases the regularity condition breaks at the point $u^1 = u^2 = 0$. As a result the first surface (1.9) gains the singularity at the origin. The second surface is non-singular despite to the breakage of the regularity condition.

Marking a regular fragment D on a surface and defining a chart mapping $\mathbf{u}^{-1}\colon U \to D$ can be treated as introducing a curvilinear coordinate system on

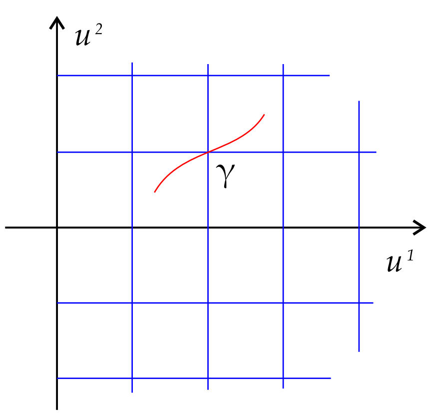

Fig. 1.1

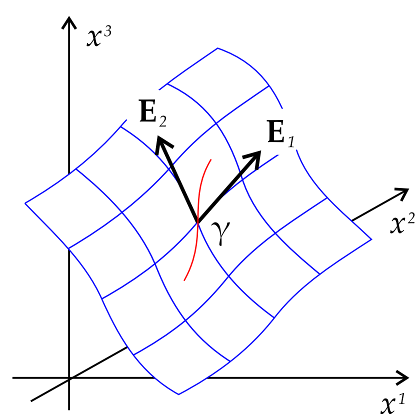

Fig. 1.2

the surface. The conditions $u^1 = \text{const}$ and $u^2 = \text{const}$ determine two families of coordinate lines on the plane of parameters u^1, u^2. They form the coordinate network in U. The mapping (1.2) maps it onto the coordinate network on the surface D (see Fig. 1.1 and Fig. 1.2). Let's consider the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ tangent to the lines of the coordinate network on the surface D:

$$\mathbf{E}_i(u^1, u^2) = \frac{\partial \mathbf{r}(u^1, u^2)}{\partial u^i}. \tag{1.10}$$

The formula (1.10) defines a pair of tangent vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ attached to each point of the surface D.

The vector-function $\mathbf{r}(u^1, u^2)$ which defines the mapping (1.2) can be written in form of the expansion in the basis of the auxiliary Cartesian coordinate system:

$$\mathbf{r}(u^1, u^2) = \sum_{q=1}^{3} x^q(u^1, u^2) \cdot \mathbf{e}_q. \tag{1.11}$$

Substituting the expansion (1.11) into (1.10) we can express the tangent vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ through the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{E}_i(u^1, u^2) = \sum_{q=1}^{3} \frac{\partial x^q(u^1, u^2)}{\partial u^i} \cdot \mathbf{e}_q. \tag{1.12}$$

Let's consider the column-vectors composed by the Cartesian coordinates of the tangent vectors $\mathbf{E}_1$ and $\mathbf{E}_2$:

$$\mathbf{E}_1 = \begin{Vmatrix} \dfrac{\partial x^1}{\partial u^1} \\[2ex] \dfrac{\partial x^2}{\partial u^1} \\[2ex] \dfrac{\partial x^3}{\partial u^1} \end{Vmatrix}, \qquad\qquad \mathbf{E}_2 = \begin{Vmatrix} \dfrac{\partial x^1}{\partial u^2} \\[2ex] \dfrac{\partial x^2}{\partial u^2} \\[2ex] \dfrac{\partial x^3}{\partial u^2} \end{Vmatrix}. \tag{1.13}$$

Note that the column-vectors (1.13) coincide with the columns in the Jacobi matrix (1.3). However, from the regularity condition (see the definition 1.1) it follows that the column of the Jacobi matrix (1.3) are linearly independent. This consideration proves the following proposition.

THEOREM 1.1. *The tangent vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ are linearly independent at each point of a surface. Therefore, they form the frame of the tangent vector fields in D.*

The frame vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ attached to some point of a surface D define the tangent plane at this point. Any vector tangent to the surface at this point lies in the tangent plane, it can be expanded in the basis formed by the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$. Let's consider some arbitrary curve γ lying completely on the surface (see Fig. 1.1 and Fig. 1.2). In parametric form such a curve is given by two functions of a parameter t. They define the curve as follows:

$$\begin{cases} u^1 = u^1(t), \\ u^2 = u^2(t). \end{cases} \tag{1.14}$$

By substituting (1.14) into (1.11) or into (1.2) we find the radius-vector of a point of the curve in the auxiliary Cartesian coordinate system $\mathbf{r}(t) = \mathbf{r}(u^1(t), u^2(t))$. Let's differentiate $\mathbf{r}(t)$ with respect to t and find the tangent vector of the curve given by the above two functions (1.14):

$$\boldsymbol{\tau}(t) = \frac{d\mathbf{r}}{dt} = \sum_{i=1}^{2} \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{du^i}{dt}.$$

Comparing this expression with (1.10), we find that $\boldsymbol{\tau}$ is expressed as follows:

$$\boldsymbol{\tau}(t)=\sum_{i=1}^{2}\dot{u}^i\cdot\mathbf{E}_i. \tag{1.15}$$

Due to (1.15) the vector $\boldsymbol{\tau}$ is a linear combination of the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ forming the tangent frame. Hence, if a curve γ lies completely on the surface, its tangent vector lies on the tangent plane to this surface, while the derivatives of the functions (1.14) are the components of the vector $\boldsymbol{\tau}$ expanded in the basis of the frame vectors $\mathbf{E}_1$ and $\mathbf{E}_2$.

§ 2. Change of curvilinear coordinates on a surface.

Let's consider two regular fragments D_1 and D_2 on some surface, each equipped with with its own curvilinear coordinate system. Assume that their intersection $D=D_1\cap D_2$ is not empty. Then in D we have two curvilinear coordinate systems u^1, u^2 and $\tilde{u}^1$, $\tilde{u}^2$. Denote by U and $\tilde{U}$ the preimages of D under the corresponding chart mappings (see Fig. 3.1 in Chapter III). Due to the bijectivity of the chart mappings (see definition 1.2) we can construct two mappings

$$\tilde{\mathbf{u}}\circ\mathbf{u}^{-1}\colon U\to\tilde{U},\qquad\qquad \mathbf{u}\circ\tilde{\mathbf{u}}^{-1}\colon \tilde{U}\to U. \tag{2.1}$$

The mappings $\tilde{\mathbf{u}}\circ\mathbf{u}^{-1}$ and $\mathbf{u}\circ\tilde{\mathbf{u}}^{-1}$ in (2.1) are also bijective, they can be represented by the following pairs of functions:

$$\begin{cases}\tilde{u}^1=\tilde{u}^1(u^1,u^2),\\ \tilde{u}^2=\tilde{u}^2(u^1,u^2).\end{cases}\qquad\qquad \begin{cases}u^1=u^1(\tilde{u}^1,\tilde{u}^2),\\ u^2=u^2(\tilde{u}^1,\tilde{u}^2).\end{cases} \tag{2.2}$$

Theorem 2.1. *The functions (2.2) representing $\tilde{\mathbf{u}}\circ\mathbf{u}^{-1}$ and $\mathbf{u}\circ\tilde{\mathbf{u}}^{-1}$ are continuously differentiable.*

Proof. We shall prove the continuous differentiability of the second pair of functions (2.2). For the first pair the proof is analogous. Let's choose some point on the chart U and map it to D. Then we choose a suitable Cartesian coordinate system in $\mathbb{E}$ such that the condition (1.5) is fulfilled and in some neighborhood of the mapped point there exists the mapping (1.7) inverse for (1.6). The mapping (1.7) is continuously differentiable.

The other curvilinear coordinate system in D induces the other pair of functions that plays the same role as the functions (1.6):

$$\begin{cases}x^1=x^1(\tilde{u}^1,\tilde{u}^2),\\ x^2=x^2(\tilde{u}^1,\tilde{u}^2).\end{cases} \tag{2.3}$$

These are two of three functions that determine the mapping $\tilde{\mathbf{u}}^{-1}$ in form of (1.2). The functions $u^1=u^1(\tilde{u}^1,\tilde{u}^2)$ and $u^2=u^2(\tilde{u}^1,\tilde{u}^2)$ that determine the mapping $\mathbf{u}\circ\tilde{\mathbf{u}}^{-1}$ in (2.2) are obtained by substituting (2.3) into the arguments of (1.7):

$$\begin{cases}u^1=u^1(x^1(\tilde{u}^1,\tilde{u}^2),x^2(\tilde{u}^1,\tilde{u}^2)),\\ u^2=u^2(x^1(\tilde{u}^1,\tilde{u}^2),x^2(\tilde{u}^1,\tilde{u}^2)).\end{cases} \tag{2.4}$$

The compositions of continuously differentiable functions in (2.4) are continuously differentiable functions. This fact completes the proof of the theorem. □

The functions (2.2), whose continuous differentiability was proved just above, perform the *transformation* or the *change* of curvilinear coordinates on a surface. They are analogous to the functions (3.5) in Chapter III.

A remark on the smoothness. If the functions (1.2) of both coordinate systems u^1, u^2 and $\tilde{u}^1, \tilde{u}^2$ belong to the smoothness class C^m, then the transition functions (2.2) also belong to the smoothness class C^m.

Let $\mathbf{r}(u^1, u^2)$ and $\mathbf{r}(\tilde{u}^1, \tilde{u}^2)$ be two vector-functions of the form (1.2) for two curvilinear coordinate systems in D. They define the mappings $\mathbf{u}^{-1}$ and $\tilde{\mathbf{u}}^{-1}$ acting from the charts U and $\tilde{U}$ to D. Due to the identity $\tilde{\mathbf{u}}^{-1} = \mathbf{u}^{-1} \circ (\mathbf{u} \circ \tilde{\mathbf{u}}^{-1})$ the function $\mathbf{r}(\tilde{u}^1, \tilde{u}^2)$ is obtained by substituting the corresponding transition functions (2.2) into the arguments of $\mathbf{r}(u^1, u^2)$:

$$\mathbf{r}(\tilde{u}^1, \tilde{u}^2) = \mathbf{r}(u^1(\tilde{u}^1, \tilde{u}^2), u^2(\tilde{u}^1, \tilde{u}^2)). \tag{2.5}$$

Let's differentiate (2.5) with respect to $\tilde{u}^j$ and take into account the chain rule and the formula (1.10) for the vectors of the tangent frame:

$$\tilde{\mathbf{E}}_j = \frac{\partial \mathbf{r}}{\partial \tilde{u}^j} = \sum_{i=1}^{2} \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial u^i}{\partial \tilde{u}^j} = \sum_{i=1}^{2} \frac{\partial u^i}{\partial \tilde{u}^j} \cdot \mathbf{E}_i.$$

Differentiating the identity $\mathbf{r}(u^1, u^2) = \mathbf{r}(\tilde{u}^1(u^1, u^2), \tilde{u}^2(u^1, u^2))$, we derive the analogous relationship inverse to the previous one:

$$\mathbf{E}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \sum_{k=1}^{2} \frac{\partial \mathbf{r}}{\partial \tilde{u}^k} \cdot \frac{\partial \tilde{u}^k}{\partial u^i} = \sum_{i=1}^{2} \frac{\partial \tilde{u}^k}{\partial u^i} \cdot \tilde{\mathbf{E}}_k.$$

It is clear that the above relationships describe the direct and inverse transitions from some tangent frame to another. Let's write them as

$$\tilde{\mathbf{E}}_j = \sum_{i=1}^{2} S^i_j \cdot \mathbf{E}_i, \qquad\qquad \mathbf{E}_i = \sum_{k=1}^{2} T^k_i \cdot \tilde{\mathbf{E}}_k, \tag{2.6}$$

where the components of the matrices S and T are given by the formulas

$$S^i_j(\tilde{u}^1, \tilde{u}^2) = \frac{\partial u^i}{\partial \tilde{u}^j}, \qquad\qquad T^k_i(u^1, u^2) = \frac{\partial \tilde{u}^k}{\partial u^i}. \tag{2.7}$$

From (2.7), we see that the transition matrices S and T are the Jacobi matrices for the mappings given by the transition functions (2.2). They are non-degenerate and are inverse to each other.

The transformations (2.2) and the transition matrices S and T related to them are used in order to construct the theory of tensors and tensor fields analogous to that which we considered in Chapter II and Chapter III. Tensors and tensor fields defined through the transformations (2.2) and transition matrices (2.7) are called *inner* tensors and *inner* tensor fields on a surface:

$$F^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum_{\substack{p_1 \ldots\, p_r \\ q_1 \ldots\, q_s}} S^{i_1}_{p_1} \ldots\, S^{i_r}_{p_r}\, T^{q_1}_{j_1} \ldots\, T^{q_s}_{j_s}\, \tilde{F}^{p_1 \ldots\, p_r}_{q_1 \ldots\, q_s}. \tag{2.8}$$

DEFINITION 2.1. An *inner tensor* of the type (r, s) on a surface is a geometric object $\mathbf{F}$ whose components in an arbitrary curvilinear coordinate system on that surface are enumerated by $(r+s)$ indices and under a change of coordinate system are transformed according to the rule (2.8).

The formula (2.8) differs from the formula (1.6) in Chapter II only in the range of indices. Each index here runs over the range of two values 1 and 2. By setting the sign factor $(-1)^S = \operatorname{sign}(\det S) = \pm 1$ into the formula (2.8) we get the definition of an inner pseudotensor

$$F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}} (-1)^S\, S^{i_1}_{p_1}\ldots\, S^{i_r}_{p_r}\, T^{q_1}_{j_1}\ldots\, T^{q_s}_{j_s}\;\tilde F^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}. \tag{2.9}$$

DEFINITION 2.3. An *inner pseudotensor* of the type (r, s) on a surface is a geometric object $\mathbf{F}$ whose components in an arbitrary curvilinear coordinate system on that surface are enumerated by $(r+s)$ indices and under a change of coordinate system are transformed according to the rule (2.9).

Inner tensorial and pseudotensorial fields are obtained by defining an inner tensor or pseudotensor at each point of a surface. The operations of addition, tensor product, contraction, transposition of indices, symmetrization and alternation for such fields are defined in a way similar to that of the case of the fields in the space $\mathbb{E}$ (see Chapter II). All properties of these operations are preserved.

A remark on the differentiation. The operation of covariant differentiation of tensor fields in the space $\mathbb{E}$ was first introduced for Cartesian coordinate systems. Then it was extended to the case of curvilinear coordinates. On surfaces, as a rule, there is no Cartesian coordinate system at all. Therefore, the operation of covariant differentiation for inner tensor fields on a surface should be defined in a somewhat different way.

§ 3. The metric tensor and the area tensor.

The choice of parameters u^1, u^2 on a surface determines the tangent frame $\mathbf{E}_1$, $\mathbf{E}_2$ on that surface. Let's consider the scalar products of the vectors $\mathbf{E}_1$, $\mathbf{E}_2$ forming the tangent frame of the surface:

$$g_{ij} = (\mathbf{E}_i\,|\,\mathbf{E}_j). \tag{3.1}$$

They compose the 2×2 Gram matrix $\mathbf{g}$ which is symmetric, non-degenerate, and positive. Therefore, we have the inequality

$$\det \mathbf{g} > 0. \tag{3.2}$$

Substituting (2.6) into (3.1), we find that under a change of a coordinate system the quantities (3.1) are transformed as the components of an inner tensorial field of the type $(0, 2)$. The tensor $\mathbf{g}$ with the components (3.1) is called the *metric tensor* of the surface. Note that the components of the metric tensor are determined by means of the scalar product in the outer space $\mathbb{E}$. Therefore, we say that the tensor field $\mathbf{g}$ *is induced* by the outer scalar product. For this reason the tensor $\mathbf{g}$ is called the metric tensor of the *induced metric.*

Symmetric tensors of the type $(0, 2)$ are related to quadratic forms. This fact yields another title for the tensor $\mathbf{g}$. It is called the *first quadratic form* of a surface. Sometimes, for the components of the first quadratic form the special notations are used: $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$. These notations are especially popular in the earlier publications on the differential geometry:

$$g_{ij} = \begin{Vmatrix} E & F \\ F & G \end{Vmatrix} \tag{3.3}$$

Since the Gram matrix $\mathbf{g}$ is non-degenerate, we can define the inverse matrix $\hat{\mathbf{g}} = \mathbf{g}^{-1}$. The components of such inverse matrix are denoted by g^{ij}, setting the indices i and j to the upper position:

$$\sum_{j=1}^{3} g^{ij}\, g_{jk} = \delta^i_j. \tag{3.4}$$

For the matrix $\hat{\mathbf{g}}$ the proposition analogous to the theorem 6.1 from Chapter II is valid. The components of this matrix define an inner tensor field of the type $(2, 0)$ on a surface, this field is called the *inverse metric tensor* or the *dual metric tensor*. The proof of this proposition is completely analogous to the proof of the theorem 6.1 in Chapter II. Therefore, here we do not give this proof.

From the symmetry of the matrix $\mathbf{g}$ and from the relationships (3.4) it follows that the components of the inverse matrix $\hat{\mathbf{g}}$ are symmetric. The direct and inverse metric tensors are used in order to lower and raise indices of tensor fields. These operations are defined by the formulas analogous to (9.1) and (9.2) in Chapter II:

$$\begin{aligned} B^{i_1\ldots\, i_{r-1}}_{j_1\ldots\, j_{s+1}} &= \sum_{k=1}^{2} A^{i_1\ldots\, i_{m-1}\, k\, i_m\ldots\, i_{r-1}}_{j_1\ldots\, j_{n-1}\, j_{n+1}\ldots\, j_{s+1}}\; g_{kj_n},\\ A^{i_1\ldots\, i_{r+1}}_{j_1\ldots\, j_{s-1}} &= \sum_{k=1}^{2} B^{i_1\ldots\, i_{m-1}\, i_{m+1}\ldots\, i_{r+1}}_{j_1\ldots\, j_{n-1}\, q\, j_n\ldots\, j_{s-1}}\; g^{qi_m}. \end{aligned} \tag{3.5}$$

The only difference of the formulas (3.5) here is that the summation index k runs over the range of two numbers 1 and 2. Due to (3.4) the operations of raising and lowering indices (3.5) are inverse to each other.

In order to define the area tensor (or the area pseudotensor) we need the following skew-symmetric 2×2 matrix:

$$d_{ij} = d^{\,ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{3.6}$$

The quantities (3.6) form the two-dimensional analog of the Levi-Civita symbol (see formula (6.8) in Chapter II). These quantities satisfy the relationship

$$\sum_{p=1}^{2}\sum_{q=1}^{2} d_{pq}\, M_{ip}\, M_{jq} = \det \mathbf{M}\; d_{ij}, \tag{3.7}$$

where $\mathbf{M}$ is some arbitrary square 2×2 matrix. The formula (3.7) is an analog of the formula (6.10) from Chapter II (see proof in [4]).

Using the quantities d_{ij} and the matrix of the metric tensor $\mathbf{g}$ in some curvilinear coordinate system, we construct the following quantities:

$$\omega_{ij} = \sqrt{\det \mathbf{g}}\; d_{ij}. \tag{3.8}$$

From (3.7) one can derive the following relationship linking the quantities ω_{ij} and $\tilde{\omega}_{pq}$ defined according to the formula (3.8) in two different coordinate systems:

$$\omega_{ij} = \operatorname{sign}(\det S)\, \sum_{p=1}^{3}\sum_{q=1}^{3} T^p_i\, T^q_j\; \tilde{\omega}_{pq}. \tag{3.9}$$

Due to (3.9) the quantities (3.8) define a skew-symmetric inner pseudotensorial field of the type $(0, 2)$. It is called the *area pseudotensor*. If on a surface D one of the two possible orientations is marked, then the formula

$$\omega_{ij} = \xi_D\, \sqrt{\det \mathbf{g}}\; d_{ij} \tag{3.10}$$

defines a tensorial field of the type $(0, 2)$. It is called the *area tensor*. The formula (3.10) differs from (3.8) only in sign factor ξ_D which is the unitary pseudoscalar field defining the orientation (compare with the formula (8.1) in Chapter II). Here one should note that not any surface admits some preferable orientation globally. The Möbius strip is a well-known example of a non-orientable surface.

§ 4. Moving frame of a surface. Veingarten's derivational formulas.

Each choice of a curvilinear coordinate system on a surface determines some frame of two tangent vectorial fields $\mathbf{E}_1, \mathbf{E}_2$ on it. The vectors of such a frame define the tangent plane at each point of the surface. However, they are insufficient for to expand an arbitrary vector of the space $\mathbb{E}$ at that point of the surface. Therefore, they are usually completed by a vector that does not belong to the tangent plane.

Definition 4.1. A *unit normal vector* $\mathbf{n}$ to a surface D at a point A is a vector of the unit length attached to the point A and perpendicular to all vectors of the tangent plane to D at that point.

The definition 4.1 fixes the unit normal vector $\mathbf{n}$ only up to the sign: at each point there are two opposite unit vectors perpendicular to the tangent plane. One of the ways to fix $\mathbf{n}$ uniquely is due to the vector product:

$$\mathbf{n} = \frac{[\mathbf{E}_1,\, \mathbf{E}_2]}{|[\mathbf{E}_1,\, \mathbf{E}_2]|}. \tag{4.1}$$

The vector $\mathbf{n}$ determined by the formula (4.1) depends on the choice of a curvilinear coordinate system. Therefore, under a change of coordinate system it can change its direction. Indeed, the relation of the frame vectors $\mathbf{E}_1, \mathbf{E}_2$ and $\tilde{\mathbf{E}}_1, \tilde{\mathbf{E}}_2$ is given by the formula (2.6). Therefore, we write

$$\mathbf{E}_1 = T^1_1 \cdot \tilde{\mathbf{E}}_1 + T^2_1 \cdot \tilde{\mathbf{E}}_2, \qquad\qquad \mathbf{E}_2 = T^1_2 \cdot \tilde{\mathbf{E}}_1 + T^2_2 \cdot \tilde{\mathbf{E}}_2.$$

Substituting these expressions into the vector product $[\mathbf{E}_1,\,\mathbf{E}_2]$, we obtain

$$[\mathbf{E}_1,\,\mathbf{E}_2] = (T^1_1\,T^2_2 - T^2_1\,T^1_2)\cdot[\tilde{\mathbf{E}}_1,\,\tilde{\mathbf{E}}_2] = \det T\cdot[\tilde{\mathbf{E}}_1,\,\tilde{\mathbf{E}}_2].$$

Now we easily derive the transformation rule for the normal vector $\mathbf{n}$:

$$\mathbf{n} = (-1)^S\cdot\tilde{\mathbf{n}}. \tag{4.2}$$

The sign factor $(-1)^S = \operatorname{sign}(\det S) = \pm 1$ here is the same as in the formula (2.8).

Another way of choosing the normal vector is possible if there is a preferable orientation on a surface. Suppose that this orientation on D is given by the unitary pseudoscalar field ξ_D. Then $\mathbf{n}$ is given by the formula

$$\mathbf{n} = \xi_D\cdot\frac{[\mathbf{E}_1,\,\mathbf{E}_2]}{|[\mathbf{E}_1,\,\mathbf{E}_2]|}. \tag{4.3}$$

In this case the transformation rule for the normal vector simplifies substantially:

$$\mathbf{n} = \tilde{\mathbf{n}}. \tag{4.4}$$

Definition 4.2. The tangent frame $\mathbf{E}_1$, $\mathbf{E}_2$ of a curvilinear coordinate system u^1, u^2 on a surface completed by the unit normal vector $\mathbf{n}$ is called the *moving frame* or the *escort frame* of this surface.

If the normal vector is chosen according to the formula (4.1), the escort frame $\mathbf{E}_1, \mathbf{E}_2, \mathbf{n}$ is always right-oriented. Therefore, in this case if we change the orientation of the tangent frame $\mathbf{E}_1$, $\mathbf{E}_2$, the direction of the normal vector $\mathbf{n}$ is changed immediately. In the other case, if $\mathbf{n}$ is determined by the formula (4.3), then its direction does not depend on the choice of the tangent frame $\mathbf{E}_1$, $\mathbf{E}_2$. This fact means that the choice of the orientation on a surface is equivalent to choosing the normal vector independent on the choice tangent vectors $\mathbf{E}_1, \mathbf{E}_2$.

There is a special case, when such an independent choice of the normal vector does exist. Let D be the boundary of a three-dimensional domain. Then one of two opposite normal vectors is the inner normal, the other is the outer normal vector. Thus, we conclude that the boundary of a three-dimensional domain in the space $\mathbb{E}$ is always orientable.

Let D be some fragment of a surface of the smoothness class C^2. The vectors of the escort frame of such a surface are continuously differentiable vector-functions of curvilinear coordinates: $\mathbf{E}_1(u^1, u^2)$, $\mathbf{E}_2(u^1, u^2)$, and $\mathbf{n}(u^1, u^2)$. The derivatives of such vectors are associated with the same point on the surface. Hence, we can write the following expansions for them:

$$\frac{\partial\mathbf{E}_j}{\partial u^i} = \sum_{k=1}^{2}\Gamma^k_{ij}\cdot\mathbf{E}_k + b_{ij}\cdot\mathbf{n}. \tag{4.5}$$

The derivatives of the unit vector $\mathbf{n}$ are perpendicular to this vector (see lemma 3.1 in Chapter I). Hence, we have the equality

$$\frac{\partial\mathbf{n}}{\partial u^i} = \sum_{k=1}^{2} c^k_i\cdot\mathbf{E}_k. \tag{4.6}$$

Let's consider the scalar product of (4.5) and the vector $\mathbf{n}$. We also consider the scalar product of (4.6) and the vector $\mathbf{E}_j$. Due to $(\mathbf{E}_k\,|\,\mathbf{n}) = 0$ we get

$$(\partial\mathbf{E}_j/\partial u^i\,|\,\mathbf{n}) = b_{ij}\,(\mathbf{n}\,|\,\mathbf{n}) = b_{ij}, \tag{4.7}$$

$$(\mathbf{E}_j\,|\,\partial\mathbf{n}/\partial u^i) = \sum_{k=1}^{2} c_i^k\,g_{kj}. \tag{4.8}$$

Let's add the left hand sides of the above formulas (4.7) and (4.8). Upon rather easy calculations we find that the sum is equal to zero:

$$(\partial\mathbf{E}_j/\partial u^i\,|\,\mathbf{n}) + (\mathbf{E}_j\,|\,\partial\mathbf{n}/\partial u^i) = \partial(\mathbf{E}_j\,|\,\mathbf{n})/\partial u^i = 0.$$

From this equality we derive the relations of b_{ij} and c_j^k in (4.5) and (4.6):

$$b_{ij} = -\sum_{k=1}^{2} c_i^k\,g_{kj}.$$

By means of the matrix of the inverse metric tensor $\hat{\mathbf{g}}$ we can invert this relationship. Let's introduce the following quite natural notation:

$$b_i^k = \sum_{j=1}^{2} b_{ij}\,g^{jk}. \tag{4.9}$$

Then the coefficients c_i^k in (4.6) can be expressed through the coefficients b_{ij} in (4.5) by means of the following formula:

$$c_i^k = -b_i^k. \tag{4.10}$$

Taking into account (4.10), we can rewrite (4.5) and (4.6) as follows:

$$\begin{aligned} \frac{\partial\mathbf{E}_j}{\partial u^i} &= \sum_{k=1}^{2} \Gamma_{ij}^k\cdot\mathbf{E}_k + b_{ij}\cdot\mathbf{n},\\ \frac{\partial\mathbf{n}}{\partial u^i} &= -\sum_{k=1}^{2} b_i^k\cdot\mathbf{E}_k. \end{aligned} \tag{4.11}$$

The expansions (4.5) and (4.6) written in form of (4.11) are called the *Veingarten's derivational formulas*. They determine the dynamics of the moving frame and play the central role in the theory of surfaces.

§ 5. Christoffel symbols and the second quadratic form.

Let's study the first Veingarten's derivational formula in two different coordinate systems u^1, u^2 and $\tilde{u}^1$, $\tilde{u}^2$ on a surface. In the coordinates u^1, u^2 it is written as

$$\frac{\partial\mathbf{E}_j}{\partial u^i} = \sum_{k=1}^{2} \Gamma_{ij}^k\cdot\mathbf{E}_k + b_{ij}\cdot\mathbf{n}. \tag{5.1}$$

In the other coordinates $\tilde{u}^1$, $\tilde{u}^2$ this formula is rewritten as

$$\frac{\partial \tilde{\mathbf{E}}_q}{\partial \tilde{u}^p} = \sum_{m=1}^{2} \tilde{\Gamma}^m_{pq} \cdot \tilde{\mathbf{E}}_m + \tilde{b}_{pq} \cdot \tilde{\mathbf{n}}. \tag{5.2}$$

Let's express the vector $\mathbf{E}_j$ in the left hand side of the formula (5.1) through the frame vectors of the second coordinate system. For this purpose we use (2.6):

$$\frac{\partial \mathbf{E}_j}{\partial u^i} = \sum_{q=1}^{2} \frac{\partial (T^q_j \cdot \tilde{\mathbf{E}}_q)}{\partial u^i} = \sum_{q=1}^{2} \frac{\partial T^q_j}{\partial u^i} \cdot \tilde{\mathbf{E}}_q + \sum_{q=1}^{2} T^q_j \cdot \frac{\partial \tilde{\mathbf{E}}_q}{\partial u^i}.$$

For the further transformation of the above expression we use the chain rule for differentiating the composite function:

$$\frac{\partial \mathbf{E}_j}{\partial u^i} = \sum_{m=1}^{2} \frac{\partial T^m_j}{\partial u^i} \cdot \tilde{\mathbf{E}}_m + \sum_{q=1}^{2} \sum_{p=1}^{2} \left(T^q_j \, \frac{\partial \tilde{u}^p}{\partial u^i} \right) \cdot \frac{\partial \tilde{\mathbf{E}}_q}{\partial \tilde{u}^p}.$$

The values of the partial derivatives $\partial \tilde{\mathbf{E}}_q / \partial \tilde{u}^p$ are determined by the formula (5.2). Moreover, we should take into account (2.7) in form of the equality $\partial \tilde{u}^p / \partial u^i = T^p_i$:

$$\begin{aligned}
\frac{\partial \mathbf{E}_j}{\partial u^i} &= \sum_{m=1}^{2} \frac{\partial T^m_j}{\partial u^i} \cdot \tilde{\mathbf{E}}_m + \sum_{q=1}^{2} \sum_{p=1}^{2} \sum_{m=1}^{2} (T^q_j \, T^p_i \, \tilde{\Gamma}^m_{pq}) \cdot \tilde{\mathbf{E}}_m + \\
&+ \sum_{q=1}^{2} \sum_{p=1}^{2} (T^q_j \, T^p_i \, \tilde{b}_{pq}) \cdot \tilde{\mathbf{n}} = \sum_{m=1}^{2} \sum_{k=1}^{2} \left(\frac{\partial T^m_j}{\partial u^i} \, S^k_m \right) \cdot \mathbf{E}_k + \\
&+ \sum_{q=1}^{2} \sum_{p=1}^{2} \sum_{m=1}^{2} \sum_{k=1}^{2} (T^q_j \, T^p_i \, \tilde{\Gamma}^m_{pq} \, S^k_m) \cdot \mathbf{E}_k + \sum_{q=1}^{2} \sum_{p=1}^{2} (T^q_j \, T^p_i \, \tilde{b}_{pq}) \cdot \tilde{\mathbf{n}}.
\end{aligned}$$

The unit normal vectors $\mathbf{n}$ and $\tilde{\mathbf{n}}$ can differ only in sign: $\mathbf{n} = \pm \tilde{\mathbf{n}}$. Hence, the above expansion for $\partial \mathbf{E}_j / \partial u^i$ and the expansion (5.1) both are the expansions in the same basis $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{n}$. Therefore, we have

$$\Gamma^k_{ij} = \sum_{m=1}^{2} S^k_m \, \frac{\partial T^m_j}{\partial u^i} + \sum_{m=1}^{2} \sum_{p=1}^{2} \sum_{q=1}^{2} S^k_m \, T^p_i \, T^q_j \; \tilde{\Gamma}^m_{pq}, \tag{5.3}$$

$$b_{ij} = \pm \sum_{p=1}^{2} \sum_{q=1}^{2} T^p_i \, T^q_j \; \tilde{b}_{pq}. \tag{5.4}$$

The formulas (5.3) and (5.4) express the transformation rules for the coefficients Γ^k_{ij} and b_{ij} in Veingarten's derivational formulas under a change of curvilinear coordinates on a surface.

In order to make certain the choice of the sign in (5.4) one should fix some rule for choosing the unit normal vector. If we choose $\mathbf{n}$ according to the formula (4.1), then under a change of coordinate system it obeys the transformation formula (4.2). In this case the equality (5.4) is written as

$$b_{ij} = \sum_{p=1}^{2} \sum_{q=1}^{2} (-1)^S \, T^p_i \, T^q_j \; \tilde{b}_{pq}. \tag{5.5}$$

It is easy to see that in this case b_{ij} are the components of an inner pseudotensorial field of the type $(0, 2)$ on a surface.

Otherwise, if we use the formula (4.3) for choosing the normal vector $\mathbf{n}$, then $\mathbf{n}$ does not depend on the choice of a curvilinear coordinate system on a surface (see formula (4.4)). In this case b_{ij} are transformed as the components of a tensorial field of the type $(0, 2)$. The formula (5.4) then takes the form

$$b_{ij} = \sum_{p=1}^{2} \sum_{q=1}^{2} T^p_i \, T^q_j \; \tilde{b}_{pq}. \tag{5.6}$$

Tensors of the type $(0, 2)$ correspond to bilinear and quadratic forms. Pseudotensors of the type $(0, 2)$ have no such interpretation. Despite to this fact the quantities b_{ij} in Veingarten's derivational formulas are called the components of the *second quadratic form* $\mathbf{b}$ of a surface. The following theorem supports this interpretation.

Theorem 5.1. *The quantities Γ^k_{ij} and b_{ij} in Veingarten's derivational formulas (4.11) are symmetric with respect to the lower indices i and j.*

Proof. In order to prove the theorem we apply the formula (1.10). Let's write this formula in the following way:

$$\mathbf{E}_j(u^1, u^2) = \frac{\partial \mathbf{r}(u^1, u^2)}{\partial u^j}. \tag{5.7}$$

Then let's substitute (5.7) into the first formula (4.11):

$$\frac{\partial^2 \mathbf{r}(u^1, u^2)}{\partial u^i \, \partial u^j} = \sum_{k=1}^{2} \Gamma^k_{ij} \cdot \mathbf{E}_k + b_{ij} \cdot \mathbf{n}. \tag{5.8}$$

The values of the mixed partial derivatives do not depend on the order of differentiation. Therefore, the left hand side of (5.8) is a vector that does not change if we transpose indices i and j. Hence, the coefficients Γ^k_{ij} and b_{ij} of its expansion in the basis $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{n}$ do not change under the transposition of the indices i and j. The theorem is proved. $\square$

Sometimes, for the components of the matrix of the second quadratic form the notations similar to (3.3) are used:

$$b_{ij} = \begin{Vmatrix} L & M \\ M & N \end{Vmatrix}. \tag{5.9}$$

These notations are especially popular in the earlier publications.

The tensor fields $\mathbf{g}$ and $\mathbf{b}$ define a pair of quadratic forms at each point of a surface. This fact explains in part their titles — the *first* and the *second* quadratic forms. The first quadratic form is non-degenerate and positive. This situation is well-known in linear algebra (see [1]). Two forms, one of which is positive, can be brought to the diagonal form simultaneously, the matrix of the positive form being brought to the unit matrix. The diagonal elements of the second quadratic form upon such diagonalization are called the *invariants of a pair of forms*. In order to calculate these invariants we consider the following contraction:

$$b_i^k = \sum_{j=1}^{2} b_{ij}\, g^{jk}. \tag{5.10}$$

The quantities b_i^k enter the second Veingarten's derivational formula (4.11). They define a tensor field (or a pseudotensorial field) of the type $(1,1)$, i. e. an operator field. The operator with the matrix (5.10) is called the *Veingarten operator*. The matrix of this operator is diagonalized simultaneously with the matrices of the first and the second quadratic forms, and its eigenvalues are exactly the invariants of that pair of forms. Let's denote them by k_1 and k_2.

Definition 5.1. The eigenvalues $k_1(u^1,u^2)$ and $k_2(u^1,u^2)$ for the matrix of the Veingarten operator are called the *principal curvatures* of a surface at its point with the coordinates u^1, u^2.

From the computational point of view the other two invariants are more convenient. These are the following ones:

$$H = \frac{k_1 + k_2}{2}, \qquad\qquad K = k_1\, k_2. \tag{5.11}$$

The invariants (5.11) can be calculated without knowing the eigenvalues of the matrix b_i^k. It is sufficient to find the trace for the matrix of the Veingarten operator and the determinant of this matrix:

$$H = \frac{1}{2}\,\mathrm{tr}(b_i^k), \qquad\qquad K = \det(b_i^k). \tag{5.12}$$

The quantity H in the formulas (5.11) and (5.12) is called the *mean curvature*, while the quantity K is called the *Gaussian curvature*. There are formulas, expressing the invariants H and K through the components of the first and the second quadratic forms (3.3) and (5.9):

$$H = \frac{1}{2}\,\frac{EN + GL - 2\,FM}{EG - F^2}, \qquad\qquad K = \frac{LN - M^2}{EG - F^2}. \tag{5.13}$$

Let $\mathbf{v}(u^1,u^2)$ and $\mathbf{w}(u^1,u^2)$ be the vectors of the basis in which the matrix of the first quadratic form is equal to the unit matrix, while the matrix of the second quadratic form is a diagonal matrix:

$$\mathbf{v} = \begin{Vmatrix} v^1 \\ v^2 \end{Vmatrix}, \qquad\qquad \mathbf{w} = \begin{Vmatrix} w^2 \\ w^2 \end{Vmatrix}. \tag{5.14}$$

Then $\mathbf{v}$ and $\mathbf{w}$ are the eigenvectors of the Veingarten operator. The vectors (5.14) have their three-dimensional realization in the space $\mathbb{E}$:

$$\mathbf{v} = v^1 \cdot \mathbf{E}_1 + v^2 \cdot \mathbf{E}_2, \qquad\qquad \mathbf{w} = w^1 \cdot \mathbf{E}_1 + w^2 \cdot \mathbf{E}_2. \tag{5.15}$$

This is the pair of the unit vectors lying on the tangent plane and being perpendicular to each other. The directions given by the vectors (5.15) are called the *principal directions* on a surface at the point with coordinates u^1, u^2. If the principal curvatures at this point are not equal to each other: $k_1 \neq k_2$, then the principal directions are determined uniquely. Otherwise, if $k_1 = k_2$, then any two mutually perpendicular directions on the tangent plane can be taken for the principal directions. A point of a surface where the principal curvatures are equal to each other ($k_1 = k_2$) is called an *umbilical point*.

A remark on the sign. Remember that depending on the way how we choose the normal vector the second quadratic form is either a tensorial field or a pseudotensorial field. The same is true for the Veingarten operator. Therefore, in general, the principal curvatures k_1 and k_2 are determined only up to the sign. The mean curvature H is also determined only up to the sign. As for the Gaussian curvature, it has no uncertainty in sign. Moreover, the sign of the Gaussian curvature divides the points of a surface into three subsets: for any point of a surface if $K > 0$, the point is called an *elliptic point*; if $K < 0$, the point is called a *hyperbolic point*; and finally, if $K = 0$, the point is called a *parabolic point*.

§ 6. Covariant differentiation of inner tensorial fields of a surface.

Let's consider the formula (5.3) and compare it with the formula (6.1) in Chapter III. These two formulas differ only in the ranges over which the indices run. Therefore the quantities Γ^k_{ij}, which appear as coefficients in the Veingarten's derivational formula, define a geometric object on a surface that is called a *connection*. The connection components Γ^k_{ij} are called the *Christoffel symbols*.

The main purpose of the Christoffel symbols Γ^k_{ij} is their usage for the covariant differentiation of tensor fields. Let's reproduce here the formula (5.12) from Chapter III for the covariant derivative modifying it for the two-dimensional case:

$$\begin{gathered}\nabla_{j_{s+1}} A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = \frac{\partial A^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial u^{j_{s+1}}} + \\ + \sum^{r}_{m=1}\sum^{2}_{v_m=1} \Gamma^{i_m}_{j_{s+1}\,v_m}\, A^{i_1\ldots\, v_m\ldots\, i_r}_{j_1\ldots\, j_s} - \sum^{s}_{n=1}\sum^{2}_{w_n=1} \Gamma^{w_n}_{j_{s+1}\,j_n}\, A^{i_1\ldots\, i_r}_{j_1\ldots\, w_n\ldots\, j_s}.\end{gathered}\tag{6.1}$$

THEOREM 6.1. *The formula (6.1) correctly defines the covariant differentiation of inner tensor fields on a surface that transforms a field of the type (r, s) into a field of the type $(r, s+1)$ if and only if the quantities Γ^k_{ij} obey the transformation rule (5.3) under a change of curvilinear coordinates on a surface.*

PROOF. Let's begin with proving the necessity. For this purpose we choose some arbitrary vector field $\mathbf{A}$ and produce the tensor field $\mathbf{B} = \nabla\mathbf{A}$ of the type $(1, 1)$ by means of the formula (6.1). The correctness of the formula (6.1) means that the components of the field $\mathbf{B}$ are transformed according to the formula (2.8). From this condition we should derive the transformation formula (5.3) for the quantities Γ^k_{ij} in (6.1). Let's write the formula (2.8) for the field $\mathbf{B} = \nabla\mathbf{A}$:

$$\frac{\partial A^k}{\partial u^i} + \sum^{2}_{j=1} \Gamma^k_{ij}\, A^j = \sum^{2}_{m=1}\sum^{2}_{p=1} S^k_m\, T^p_i \left(\frac{\partial \tilde{A}^m}{\partial \tilde{u}^p} + \sum^{2}_{q=1} \tilde{\Gamma}^m_{pq}\, \tilde{A}^q\right).$$

Then we expand the brackets in the right hand side of this relationship. In the first summand we replace T^p_i by $\partial\tilde{u}^p/\partial u^i$ according to the formula (2.7) and we express $\tilde{A}^m$ through A^j according to the transformation rule for a vector field:

$$\begin{gathered}\sum_{p=1}^{2} T^p_i\,\frac{\partial\tilde{A}^m}{\partial\tilde{u}^p}=\sum_{p=1}^{2}\frac{\partial\tilde{u}^p}{\partial u^i}\,\frac{\partial\tilde{A}^m}{\partial\tilde{u}^p}=\frac{\partial\tilde{A}^m}{\partial u^i}=\\ =\frac{\partial}{\partial u^i}\left(\sum_{k=1}^{2}T^m_k\,A^k\right)=\sum_{j=1}^{2}\frac{\partial T^m_j}{\partial u^i}\,A^j+\sum_{j=1}^{2}T^m_j\,\frac{\partial A^j}{\partial u^i}.\end{gathered}\tag{6.2}$$

Taking into account (6.2), we can cancel the partial derivatives in the previous equality and bring it to the following form:

$$\sum_{j=1}^{2}\Gamma^k_{ij}\,A^j=\sum_{j=1}^{2}\sum_{m=1}^{2}S^k_m\,\frac{\partial T^m_j}{\partial u^i}\,A^j+\sum_{m=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2}S^k_m\,T^p_i\,\tilde{\Gamma}^m_{pq}\,\tilde{A}^q.$$

Now we need only to express $\tilde{A}^q$ through A^j applying the transformation rule for the components of a vectorial field and then extract the common factor A^j:

$$\sum_{j=1}^{2}\left(\Gamma^k_{ij}-\sum_{m=1}^{2}S^k_m\,\frac{\partial T^m_j}{\partial u^i}-\sum_{m=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2}S^k_m\,T^p_i\,T^q_j\,\tilde{\Gamma}^m_{pq}\right)A^j=0.$$

Since $\mathbf{A}$ is an arbitrary vector field, each coefficient enclosed into round brackets in the above sum should vanish separately. This condition coincides exactly with the transformation rule (5.3). Thus, the necessity is proved.

Let's proceed with proving the sufficiency. Suppose that the condition (5.3) is fulfilled. Let's choose some tensorial field $\mathbf{A}$ of the type (r,s) and prove that the quantities $\nabla_{j_{s+1}}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ determined by the formula (6.1) are transformed as the components of a tensorial field of the type $(r,s+1)$. Looking at the formula (6.1) we see that it contains the partial derivative $\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}/\partial u^{j_{s+1}}$ and other $r+s$ terms. Let's denote these terms by $A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]$ and $A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}0\\n\end{smallmatrix}\right]$. Then

$$\begin{gathered}\nabla_{j_{s+1}}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}/\partial u^{j_{s+1}}+\\ +\sum_{m=1}^{r}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]-\sum_{n=1}^{s}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}0\\n\end{smallmatrix}\right].\end{gathered}\tag{6.3}$$

The tensorial nature of $\mathbf{A}$ means that its components are transformed according to the formula (2.8). Therefore, in the first term of (6.3) we get:

$$\begin{gathered}\frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial u^{j_{s+1}}}=\frac{\partial}{\partial u^{j_{s+1}}}\left(\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}}S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}\right)=\\ =\sum_{\substack{p_1\ldots\,p_r\\ q_1\ldots\,q_s}}\sum_{q_{s+1}=1}^{2}S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\,\frac{\partial\tilde{u}^{q_{s+1}}}{\partial u^{j_{s+1}}}\,\frac{\partial\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}}{\partial\tilde{u}^{q_{s+1}}}+\end{gathered}$$

$$+\sum_{m=1}^{r}\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots\frac{\partial S^{i_m}_{p_m}}{\partial u^{j_{s+1}}}\ldots S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}+$$

$$+\sum_{n=1}^{s}\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots\frac{\partial T^{q_n}_{j_n}}{\partial u^{j_{s+1}}}\ldots T^{q_s}_{j_s}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}.$$

Here we used the Leibniz rule for differentiating the product of multiple functions and the chain rule in order to express the derivatives with respect to $u^{j_{s+1}}$ through the derivatives with respect to $\tilde{u}^{q_{s+1}}$. For the further transformation of the above formula we replace $\partial\tilde{u}^{q_{s+1}}/\partial u^{j_{s+1}}$ by $T^{q_{s+1}}_{j_{s+1}}$ according to (2.7) and we use the following identities based on the fact that S and T are mutually inverse matrices:

$$\begin{aligned}\frac{\partial S^{i_m}_{p_m}}{\partial u^{j_{s+1}}}&=-\sum_{v_m=1}^{2}\sum_{k=1}^{2}S^{i_m}_{v_m}\,\frac{\partial T^{v_m}_{k}}{\partial u^{j_{s+1}}}\,S^{k}_{p_m},\\ \frac{\partial T^{q_n}_{j_n}}{\partial u^{j_{s+1}}}&=\sum_{w_n=1}^{2}\sum_{k=1}^{2}T^{w_n}_{j_n}\,S^{k}_{w_n}\,\frac{\partial T^{q_n}_{k}}{\partial u^{j_{s+1}}}.\end{aligned}\tag{6.4}$$

Upon substituting (6.4) into the preceding equality it is convenient to transpose the summation indices: p_m with v_m and q_n with w_n. Then we get

$$\begin{aligned}\frac{\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial u^{j_{s+1}}}&=\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\times\\ &\times\left(\sum_{q_{s+1}=1}^{2}T^{q_{s+1}}_{j_{s+1}}\,\frac{\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}}{\partial\tilde{u}^{q_{s+1}}}-\sum_{m=1}^{r}V\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]+\sum_{n=1}^{s}W\left[\begin{smallmatrix}0\\n\end{smallmatrix}\right]\right),\end{aligned}\tag{6.5}$$

where the following notations are used for the sake of simplicity:

$$\begin{aligned}V\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]&=\sum_{v_m=1}^{2}\sum_{k=1}^{2}S^{k}_{v_m}\,\frac{\partial T^{p_m}_{k}}{\partial u^{j_{s+1}}}\,\tilde{A}^{p_1\ldots\,v_m\ldots\,p_r}_{q_1\ldots\,q_s},\\ W\left[\begin{smallmatrix}0\\n\end{smallmatrix}\right]&=\sum_{w_n=1}^{2}\sum_{k=1}^{2}S^{k}_{q_n}\,\frac{\partial T^{w_n}_{k}}{\partial u^{j_{s+1}}}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,w_n\ldots\,q_s}.\end{aligned}\tag{6.6}$$

No let's consider $A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]$ in (6.3). They are obviously defined as follows:

$$A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]=\sum_{v_m=1}^{2}\Gamma^{i_m}_{j_{s+1}\,v_m}\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}.$$

Applying the transformation rule (2.8) to the components of the field **A**, we get:

$$\begin{aligned}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\left[\begin{smallmatrix}m\\0\end{smallmatrix}\right]&=\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}}\sum_{v_m=1}^{2}S^{i_1}_{p_1}\ldots S^{v_m}_{p_m}\ldots S^{i_r}_{p_r}\times\\ &\times T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\,\Gamma^{i_m}_{j_{s+1}\,v_m}\,\tilde{A}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}.\end{aligned}\tag{6.7}$$

For the further transformation of the above expression we use (5.3) written as

$$\Gamma^{i_m}_{j_{s+1}\,v_m}=\sum_{k=1}^{2}S^{i_m}_k\,\frac{\partial T^k_{v_m}}{\partial u^{j_{s+1}}}+\sum_{k=1}^{2}\sum_{q=1}^{2}\sum_{q_{s+1}=1}^{2}S^{i_m}_k\,T^{q_{s+1}}_{j_{s+1}}\,T^q_{v_m}\,\tilde\Gamma^k_{q_{s+1}\,q}.$$

Immediately after substituting this expression into (6.7) we perform the cyclic transposition of the summation indices: $r\to p_m\to v_m\to r$. Some sums in the resulting expression are evaluated explicitly if we take into account the fact that the transition matrices S and T are inverse to each other:

$$\begin{aligned}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\begin{bmatrix}m\\0\end{bmatrix}&=\sum_{\substack{p_1\ldots\,p_r\\q_1\ldots\,q_s}}S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\times\\&\times\left(\sum_{q_{s+1}=1}^{2}T^{q_{s+1}}_{j_{s+1}}\,\tilde A^{p_1\ldots\,p_r}_{q_1\ldots\,q_{s+1}}\begin{bmatrix}m\\0\end{bmatrix}+V\begin{bmatrix}m\\0\end{bmatrix}\right).\end{aligned}\tag{6.8}$$

By means of the analogous calculations one can derive the following formula:

$$\begin{aligned}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_{s+1}}\begin{bmatrix}0\\n\end{bmatrix}&=\sum_{\substack{p_1\ldots\,p_r\\q_1\ldots\,q_s}}S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}\,T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\times\\&\times\left(\sum_{q_{s+1}=1}^{2}T^{q_{s+1}}_{j_{s+1}}\,\tilde A^{p_1\ldots\,p_r}_{q_1\ldots\,q_{s+1}}\begin{bmatrix}0\\n\end{bmatrix}+W\begin{bmatrix}0\\n\end{bmatrix}\right).\end{aligned}\tag{6.9}$$

Now we substitute (6.5), (6.8), and (6.9) into the formula (6.3). Then the entries of $V\begin{bmatrix}m\\0\end{bmatrix}$ and $W\begin{bmatrix}0\\n\end{bmatrix}$ do cancel each other. A a residue, upon collecting the similar terms and cancellations, we get the formula expressing the transformation rule (2.8) applied to the components of the field $\nabla\mathbf{A}$. The theorem 6.1 is proved. □

The theorem 6.1 yields a universal mechanism for constructing the covariant differentiation. It is sufficient to have a connection whose components are transformed according to the formula (5.3). We can compare two connections: the Euclidean connection in the space $\mathbb{E}$ constructed by means of the Cartesian coordinates and a connection on a surface whose components are given by the Veingarten's derivational formulas. Despite to the different origin of these two connections, the covariant derivatives defined by them have many common properties. It is convenient to formulate these properties using covariant derivatives along vector fields. Let $\mathbf{X}$ be a vector field on a surface. For any tensor field $\mathbf{A}$ of the type (r,s) we define the tensor field $\mathbf{B}=\nabla_{\mathbf{X}}\mathbf{A}$ of the same type (r,s) given by the following formula:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{q=1}^{2}X^q\,\nabla_qA^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}.\tag{6.10}$$

Theorem 6.2. *The operation of covariant differentiation of tensor fields possesses the following properties:*

(1) $\nabla_{\mathbf{X}}(\mathbf{A}+\mathbf{B})=\nabla_{\mathbf{X}}\mathbf{A}+\nabla_{\mathbf{X}}\mathbf{B}$;
(2) $\nabla_{\mathbf{X}+\mathbf{Y}}\mathbf{A}=\nabla_{\mathbf{X}}\mathbf{A}+\nabla_{\mathbf{Y}}\mathbf{A}$;
(3) $\nabla_{\xi\cdot\mathbf{X}}\mathbf{A}=\xi\cdot\nabla_{\mathbf{X}}\mathbf{A}$;
(4) $\nabla_{\mathbf{X}}(\mathbf{A}\otimes\mathbf{B})=\nabla_{\mathbf{X}}\mathbf{A}\otimes\mathbf{B}+\mathbf{A}\otimes\nabla_{\mathbf{X}}\mathbf{B}$;
(5) $\nabla_{\mathbf{X}}C(\mathbf{A})=C(\nabla_{\mathbf{X}}\mathbf{A})$;

where $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary differentiable tensor fields, while* $\mathbf{X}$ *and* $\mathbf{Y}$ *are arbitrary vector fields and* ξ *is an arbitrary scalar field.*

Looking attentively at the theorem 6.2 and at the formula (6.10), we see that the theorem 6.2 is a copy of the theorem 5.2 from Chapter II, while the formula (6.10) is a two-dimensional analog of the formula (5.5) from the same Chapter II. However, the proof there is for the case of the Euclidean connection in the space $\mathbb{E}$. Therefore we need to give another proof.

Proof. Let's choose some arbitrary curvilinear coordinate system on a surface and prove the theorem by means of direct calculations in coordinates. Denote $\mathbf{C} = \mathbf{A} + \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are two tensorial fields of the type (r, s). Then

$$C^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} + B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}.$$

Substituting $C^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ into (6.1), upon rather simple calculations we get

$$\nabla_{j_{s+1}} C^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \nabla_{j_{s+1}} A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} + \nabla_{j_{s+1}} B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}.$$

The rest is to multiply both sides of the above equality by $X^{j_{s+1}}$ and perform summation over the index j_{s+1}. Applying (6.10), we derive the formula of the item (1) in the theorem.

Note that the quantities $B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ in the formula (6.10) are obtained as the linear combinations of the components of $\mathbf{X}$. The items (2) and (3) of the theorem follow immediately from this fact.

Let's proceed with the item (4). Denote $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$. Then for the components of the tensor field $\mathbf{C}$ we have the equality

$$C^{i_1\ldots\,i_{r+p}}_{j_1\ldots\,j_{s+q}} = A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}. \tag{6.11}$$

Let's substitute the quantities $C^{i_1\ldots\,i_{r+p}}_{j_1\ldots\,j_{s+q}}$ from (6.11) into the formula (6.1) defining the covariant derivative. As a result for $\mathbf{D} = \nabla\mathbf{C}$ we derive

$$\begin{aligned}
D^{i_1\ldots\,i_{r+p}}_{j_1\ldots\,j_{s+q+1}} &= \left(\partial A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}/\partial u^{j_{s+q+1}}\right)\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}} + \\
&+ A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\left(\partial B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}/\partial u^{j_{s+q+1}}\right) + \\
&+ \sum_{m=1}^{r}\sum_{v_m=1}^{2}\Gamma^{i_m}_{j_{s+q+1}\,v_m}\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}} + \\
&+ \sum_{m=r+1}^{r+p}\sum_{v_m=1}^{2} A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\Gamma^{i_m}_{j_{s+q+1}\,v_m}\,B^{i_{r+1}\ldots\,v_m\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}} - \\
&- \sum_{n=1}^{s}\sum_{w_n=1}^{2}\Gamma^{w_n}_{j_{s+q+1}\,j_n}\,A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}} - \\
&- \sum_{n=s+1}^{s+q}\sum_{w_n=1}^{2} A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\Gamma^{w_n}_{j_{s+q+1}\,j_n}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,w_n\ldots\,j_{s+q}}.
\end{aligned}$$

Note that upon collecting the similar terms the above huge formula can be transformed to the following shorter one:

$$\begin{gathered}\nabla_{j_{s+q+1}}\bigl(A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}\bigr)=\bigl(\nabla_{j_{s+q+1}}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\bigr)\times\\ \times\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}+A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\bigl(\nabla_{j_{s+q+1}}B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}\bigr).\end{gathered}\tag{6.12}$$

Now in order to prove the fourth item of the theorem it is sufficient to multiply (6.12) by $X^{j_{s+q+1}}$ and sum up over the index j_{s+q+1}.

Proceeding with the last fifth item of the theorem, we consider two tensorial fields $\mathbf{A}$ and $\mathbf{B}$ one of which is the contraction of another:

$$B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{k=1}^{2}A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}.\tag{6.13}$$

Substituting (6.13) into (6.1), for $\nabla_{j_{s+1}}B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ we obtain

$$\begin{gathered}\nabla_{j_{s+1}}B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{k=1}^{2}\frac{\partial A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}}{\partial u^{j_{s+1}}}\,+\\ +\sum_{m=1}^{r}\sum_{k=1}^{2}\sum_{v_m=1}^{2}\Gamma^{i_m}_{j_{s+1}\,v_m}\,A^{i_1\ldots\,v_m\ldots\,k\,\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}\,-\\ -\sum_{n=1}^{s}\sum_{k=1}^{2}\sum_{w_n=1}^{2}\Gamma^{w_n}_{j_{s+1}\,j_n}\,A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,k\,\ldots\,j_s}.\end{gathered}\tag{6.14}$$

The index v_m in (6.14) can be either to the left of the index k or to the right of it. The same is true for the index w_n. However, the formula (6.14) does not comprise the terms where v_m or w_n replaces the index k. Such terms would have the form:

$$\sum_{k=1}^{2}\sum_{v=1}^{2}\Gamma^{k}_{j_{s+1}\,v}\,\;A^{i_1\ldots\,i_{p-1}\,v\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s},\tag{6.15}$$

$$-\sum_{k=1}^{2}\sum_{w=1}^{2}\Gamma^{w}_{j_{s+1}\,k}\,\;A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,w\,j_q\ldots\,j_s}.\tag{6.16}$$

It is easy to note that (6.15) and (6.16) differ only in sign. It is sufficient to rename k to v and w to k in the formula (6.16). Adding both (6.15) and (6.16) to (6.14) would not break the equality. But upon adding them one can rewrite the equality (6.14) in the following form:

$$\nabla_{j_{s+1}}B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{k=1}^{2}\nabla_{j_{s+1}}A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}.\tag{6.17}$$

No in order to complete the proof of the item (5), and thus prove the theorem in whole, it is sufficient to multiply the equality (6.17) by $X^{j_{s+1}}$ and sum up over the index j_{s+1}. $\square$

Among the five properties of the covariant derivative listed in the theorem 6.2 the fourth property written as (6.12) is most often used in calculations. Let's rewrite the equality (6.12) as follows:

$$\nabla_k\bigl(A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,B^{v_1\ldots\,v_p}_{w_1\ldots\,w_q}\bigr)=\bigl(\nabla_k A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\bigr)\,B^{v_1\ldots\,v_p}_{w_1\ldots\,w_q}+A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\bigl(\nabla_k B^{v_1\ldots\,v_p}_{w_1\ldots\,w_q}\bigr). \tag{6.18}$$

The formula (6.18) is produced from (6.12) simply by renaming the indices; however, it is more convenient for reception.

§ 7. Concordance of metric and connection on a surface.

Earlier we have noted that the covariant differential of the metric tensor $\mathbf{g}$ in the Euclidean space $\mathbb{E}$ is equal to zero (see formula (6.7) in Chapter II). This property was called the concordance of the metric and connection. Upon passing to curvilinear coordinates we used this property in order to express the components of the Euclidean connection Γ^k_{ij} through the components of the metric tensor (see formula (7.8) in Chapter III). Let's study whether the metric and the connection on surfaces are concordant or not. The answer here is also positive. It is given by the following theorem.

Theorem 7.1. *The components of the metric tensor g^{ij} and the connection components Γ^k_{ij} in arbitrary coordinates on a surface are related by the equality*

$$\nabla_k g_{ij}=\frac{\partial g_{ij}}{\partial u^k}-\sum_{q=1}^{2}\Gamma^q_{ki}\,g_{qj}-\sum_{q=1}^{2}\Gamma^q_{kj}\,g_{iq}=0 \tag{7.1}$$

which expresses the concordance condition for the metric and connection.

Proof. Let's consider the first Veingarten's derivational formula in (4.11) and let's rewrite it renaming some indices:

$$\frac{\partial\mathbf{E}_i}{\partial u^k}=\sum_{q=1}^{2}\Gamma^q_{ki}\cdot\mathbf{E}_q+b_{ki}\cdot\mathbf{n}. \tag{7.2}$$

Let's take the scalar products of both sides of (7.2) by $\mathbf{E}_j$ and remember that the vectors $\mathbf{E}_j$ and $\mathbf{n}$ are perpendicular. The scalar product of $\mathbf{E}_q$ and $\mathbf{E}_j$ in the right hand side yields the element g_{qj} of the Gram matrix:

$$(\partial\mathbf{E}_i/\partial u^k\,|\,\mathbf{E}_j)=\sum_{q=1}^{2}\Gamma^q_{ki}\,g_{qj}. \tag{7.3}$$

Now let's transpose the indices i and j in (7.3) and take into account the symmetry of the Gram matrix. As a result we obtain

$$(\mathbf{E}_i\,|\,\partial\mathbf{E}_j/\partial u^k)=\sum_{q=1}^{2}\Gamma^q_{kj}\,g_{iq}. \tag{7.4}$$

Then let's add (7.3) with (7.4) and remember the Leibniz rule as applied to the differentiation of the scalar product in the space $\mathbb{E}$:

$$(\partial \mathbf{E}_i/\partial u^k \,|\, \mathbf{E}_j) + (\mathbf{E}_i \,|\, \partial \mathbf{E}_j/\partial u^k) = \partial(\mathbf{E}_i \,|\, \mathbf{E}_j)/\partial u^k =$$

$$= \partial g_{ij}/\partial u^k = \sum_{q=1}^{2} \Gamma^q_{ki}\, g_{qj} + \sum_{q=1}^{2} \Gamma^q_{kj}\, g_{iq}.$$

Now it is easy to see that the equality just obtained coincides in essential with (7.1). The theorem is proved. □

As an immediate consequence of the theorems 7.1 and 5.1 we get the following formula for the connection components:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{r=1}^{2} g^{kr} \left(\frac{g_{rj}}{\partial u^i} + \frac{g_{ir}}{\partial u^j} - \frac{g_{ij}}{\partial u^r} \right). \tag{7.5}$$

We do not give its prove here since, in essential, it is the same as in the case of the formula (7.8) in Chapter III.

From the condition $\nabla_q g_{ij} = 0$ one can easily derive that the covariant derivatives of the inverse metric tensor are also equal to zero. For this purpose one should apply the formula (3.4). The covariant derivatives of the identical operator field with the components δ^i_k are equal to zero. Indeed, we have

$$\nabla_q \delta^i_k = \frac{\partial(\delta^i_k)}{\partial u^q} + \sum_{r=1}^{2} \Gamma^i_{qr} \delta^r_k - \sum_{r=1}^{2} \Gamma^r_{qk} \delta^i_r = 0. \tag{7.6}$$

Let's differentiate both sides of (3.4) and take into account (7.6):

$$\nabla_q \left(\sum_{j=1}^{2} g^{ij}\, g_{jk} \right) = \sum_{j=1}^{2} (\nabla_q g^{ij}\, g_{jk} + g^{ij}\, \nabla_q g_{jk}) =$$

$$= \sum_{j=1}^{2} \nabla_q g^{ij}\, g_{jk} = \nabla_q \delta^i_k = 0. \tag{7.7}$$

In deriving (7.7) we used the items (4) and (5) from the theorem 6.2. The procedure of lowering j by means of the contraction with the metric tensor g_{jk} is an invertible operation. Therefore, (7.7) yields $\nabla_q g^{ij} = 0$. Now the concordance condition for the metric and connection is written as a pair of two relationships

$$\nabla \mathbf{g} = 0, \qquad \nabla \hat{\mathbf{g}} = 0, \tag{7.8}$$

which look exactly like the relationships (6.7) in Chapter II for the case of metric tensors in the space $\mathbb{E}$.

Another consequence of the theorem 7.1 is that the index raising and the index lowering operations performed by means of contractions with the metric tensors

$\hat{\mathbf{g}}$ and $\mathbf{g}$ commute with the operations of covariant differentiations. This fact is presented by the following two formulas:

$$\begin{aligned}\nabla_q\left(\sum_{k=1}^{2} g_{ik}\,A^{\dots\,k\,\dots}_{\dots\;\;\dots}\right) &= \sum_{k=1}^{2} g_{ik}\,\nabla_q A^{\dots\,k\,\dots}_{\dots\;\;\dots},\\ \nabla_q\left(\sum_{k=1}^{2} g^{ik}\,A^{\dots\;\;\dots}_{\dots\,k\,\dots}\right) &= \sum_{k=1}^{2} g^{ik}\,\nabla_q A^{\dots\;\;\dots}_{\dots\,k\,\dots}.\end{aligned} \tag{7.9}$$

The relationship (7.9) is easily derived from (7.8) using the items (4) and (5) in the theorem 6.2.

Theorem 7.2. *The covariant differential of the area pseudotensor* (3.8) *on any surface is equal to zero:* $\nabla\boldsymbol{\omega} = 0$.

In order to prove this theorem we need two auxiliary propositions which are formulated as the following lemmas.

Lemma 7.1. *For any square matrix M whose components are differentiable functions of some parameter x the equality*

$$\frac{d(\ln\det M)}{dx} = \operatorname{tr}(M^{-1}\,M') \tag{7.10}$$

is fulfilled, where M' is the square matrix composed by the derivatives of the corresponding components of the matrix M.

Lemma 7.2. *For any square 2×2 matrix M the equality*

$$\sum_{q=1}^{2}\bigl(M_{iq}\,d_{qj} + M_{jq}\,d_{iq}\bigr) = \operatorname{tr} M\,d_{ij} \tag{7.11}$$

is fulfilled, where d_{ij} are the components of the skew-symmetric matrix determined by the relationship (3.6).

The proof of these two lemmas 7.1 and 7.2 as well as the proof of the above formula (3.7) from § 3 can be found in [4].

Let's apply the lemma 7.1 to the matrix of the metric tensor. Let $x = u^k$. Then we rewrite the relationship (7.10) as follows:

$$\frac{1}{\sqrt{\det\mathbf{g}}}\,\frac{\partial\sqrt{\det\mathbf{g}}}{\partial u^k} = \frac{1}{2}\sum_{q=1}^{2}\sum_{p=1}^{2} g^{qp}\,\frac{\partial g_{qp}}{\partial u^k}. \tag{7.12}$$

Note that in (7.11) any array of four numbers enumerated with two indices can play the role of the matrix M. Having fixed the index k, one can use the connection components Γ^j_{ki} as such an array. Then we obtain

$$\sum_{q=1}^{2}\bigl(\Gamma^q_{ki}\,d_{qj} + \Gamma^q_{kj}\,d_{iq}\bigr) = \sum_{q=1}^{2}\Gamma^q_{kq}\,d_{ij}. \tag{7.13}$$

Proof for the theorem 7.2. The components of the area pseudotensor $\boldsymbol{\omega}$ are determined by the formula (3.8). In order to find the components of the pseudotensor $\nabla\boldsymbol{\omega}$ we apply the formula (6.1). It yields

$$\begin{aligned}\nabla_k\omega_{ij} &= \frac{\partial\sqrt{\det\mathbf{g}}}{\partial u^k}\,d_{ij} - \sum_{q=1}^{2}\sqrt{\det\mathbf{g}}\,\bigl(\Gamma^q_{ki}\,d_{qj} + \Gamma^q_{kj}\,d_{iq}\bigr) = \\ &= \sqrt{\det\mathbf{g}}\left(\frac{1}{\sqrt{\det\mathbf{g}}}\,\frac{\partial\sqrt{\det\mathbf{g}}}{\partial u^k}\,d_{ij} - \sum_{q=1}^{2}\bigl(\Gamma^q_{ki}\,d_{qj} + \Gamma^q_{kj}\,d_{iq}\bigr)\right).\end{aligned}$$

For the further transformation of this expression we apply (7.12) and (7.13):

$$\nabla_k\omega_{ij} = \sqrt{\det\mathbf{g}}\left(\frac{1}{2}\sum_{q=1}^{2}\sum_{p=1}^{2} g^{qp}\,\frac{\partial g_{qp}}{\partial u^k} - \sum_{q=1}^{2}\Gamma^q_{kq}\right) d_{ij}. \tag{7.14}$$

Now let's express Γ^q_{kq} through the components of the metric tensor by means of the formula (7.5). Taking into account the symmetry of g^{pq}, we get

$$\sum_{q=1}^{2}\Gamma^q_{kq} = \frac{1}{2}\sum_{q=1}^{2}\sum_{p=1}^{2} g^{qp}\left(\frac{g_{pq}}{\partial u^k} + \frac{g_{kp}}{\partial u^q} - \frac{g_{kq}}{\partial u^p}\right) = \frac{1}{2}\sum_{q=1}^{2}\sum_{p=1}^{2} g^{qp}\,\frac{g_{qp}}{\partial u^k}.$$

Substituting this expression into the formula (7.14), we find that it vanishes. Hence, $\nabla_k\omega_{ij} = 0$. The theorem is proved. □

A remark on the sign. The area tensor differs from the area pseudotensor only by the scalar sign factor ξ_D. Therefore, the proposition of the theorem 7.2 for the area tensor of an arbitrary surface is also valid.

A remark on the dimension. For the volume tensor (and for the volume pseudotensor) in the Euclidean space $\mathbb{E}$ we have the analogous proposition: it states that $\nabla\boldsymbol{\omega} = 0$. The proof of this proposition is even more simple than the proof of the theorem 7.2. The components of the field $\boldsymbol{\omega}$ in any Cartesian coordinate system in $\mathbb{E}$ are constants. Hence, their derivatives are zero.

§ 8. Curvature tensor.

The covariant derivatives in the Euclidean space $\mathbb{E}$ are reduced to the partial derivatives in any Cartesian coordinates. Is there such a coordinate system for covariant derivatives on a surface? The answer to this question is related to the commutators. Let's choose a vector field $\mathbf{X}$ and calculate the tensor field $\mathbf{Y}$ of the type $(1, 2)$ with the following components:

$$Y^k_{ij} = \nabla_i\nabla_j X^k - \nabla_j\nabla_i X^k. \tag{8.1}$$

In order to calculate the components of the field $\mathbf{Y}$ we apply the formula (6.1):

$$\begin{aligned}\nabla_i\nabla_j X^k &= \frac{\partial(\nabla_j X^k)}{\partial u^i} + \sum_{q=1}^{2}\Gamma^k_{iq}\nabla_j X^q - \sum_{q=1}^{2}\Gamma^q_{ij}\nabla_q X^k,\\ \nabla_j\nabla_i X^k &= \frac{\partial(\nabla_i X^k)}{\partial u^j} + \sum_{q=1}^{2}\Gamma^k_{jq}\nabla_i X^q - \sum_{q=1}^{2}\Gamma^q_{ji}\nabla_q X^k.\end{aligned} \tag{8.2}$$

Let's subtract the second relationship (8.2) from the first one. Then the last terms in them do cancel each other due to the symmetry of Γ^k_{ij}:

$$
\begin{aligned}
Y^k_{ij} &= \frac{\partial}{\partial u^i}\left(\frac{\partial X^k}{\partial u^j}+\sum_{r=1}^{2}\Gamma^k_{jr}\,X^r\right)-\frac{\partial}{\partial u^j}\left(\frac{\partial X^k}{\partial u^i}+\sum_{r=1}^{2}\Gamma^k_{ir}\,X^r\right)+\\
&+\sum_{q=1}^{2}\Gamma^k_{iq}\left(\frac{\partial X^q}{\partial u^j}+\sum_{r=1}^{2}\Gamma^q_{jr}\,X^r\right)-\sum_{q=1}^{2}\Gamma^k_{jq}\left(\frac{\partial X^q}{\partial u^i}+\sum_{r=1}^{2}\Gamma^q_{ir}\,X^r\right).
\end{aligned}
$$

Upon expanding the brackets and some cancellations we get

$$
Y^k_{ij}=\sum_{r=1}^{2}\left(\frac{\partial\Gamma^k_{jr}}{\partial u^i}-\frac{\partial\Gamma^k_{ir}}{\partial u^j}+\sum_{q=1}^{2}\bigl(\Gamma^k_{iq}\Gamma^q_{jr}-\Gamma^k_{jq}\Gamma^q_{ir}\bigr)\right)X^r. \tag{8.3}
$$

It is important to note that the formula (8.3) does not contain the derivatives of the components of $\mathbf{X}$ — they are canceled. Let's denote

$$
R^k_{rij}=\frac{\partial\Gamma^k_{jr}}{\partial u^i}-\frac{\partial\Gamma^k_{ir}}{\partial u^j}+\sum_{q=1}^{2}\Gamma^k_{iq}\Gamma^q_{jr}-\sum_{q=1}^{2}\Gamma^k_{jq}\Gamma^q_{ir}. \tag{8.4}
$$

The formula (8.3) for the components of the field (8.1) then can be written as

$$
(\nabla_i\nabla_j-\nabla_j\nabla_i)X^k=\sum_{r=1}^{2}R^k_{rij}\,X^r. \tag{8.5}
$$

Let's replace the vector field $\mathbf{X}$ by a covector field. Performing the similar calculations, in this case we obtain

$$
(\nabla_i\nabla_j-\nabla_j\nabla_i)X_k=-\sum_{r=1}^{2}R^r_{kij}\,X_r. \tag{8.6}
$$

The formulas (8.5) and (8.6) can be generalized for the case of an arbitrary tensor field $\mathbf{X}$ of the type (r,s):

$$
\begin{aligned}
(\nabla_i\nabla_j-\nabla_j\nabla_i)X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \sum_{m=1}^{r}\sum_{v_m=1}^{2}R^{i_m}_{v_m\,ij}\,X^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}-\\
&-\sum_{n=1}^{s}\sum_{w_n=1}^{2}R^{w_n}_{j_n\,ij}\,X^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}.
\end{aligned} \tag{8.7}
$$

Comparing (8.5), (8.6), and (8.7), we see that all of them contain the quantities R^k_{rij} given by the formula (8.4).

THEOREM 8.1. *The quantities R^k_{rij} introduced by the formula* (8.4) *define a tensor field of the type* $(1,3)$. *This tensor field is called the curvature tensor or the Riemann tensor.*

The theorem 8.1 can be proved directly on the base of the formula (5.3). However, we give another proof which is more simple.

LEMMA 8.1. *Let* $\mathbf{R}$ *be a geometric object which is presented by a four-dimensional array* R^k_{rij} *in coordinates. If the contraction of* $\mathbf{R}$ *with an arbitrary vector* $\mathbf{X}$

$$Y^k_{ij} = \sum_{q=1}^{2} R^k_{qij}\,X^q \tag{8.8}$$

is a tensor of the type $(1,2)$, *then the object* $\mathbf{R}$ *itself is a tensor of the type* $(1,3)$.

PROOF OF THE LEMMA. Let u^1, u^2 and $\tilde u^1$, $\tilde u^2$ be two curvilinear coordinate systems on a surface. Let's fix some numeric value of the index r ($r=1$ or $r=2$). Since $\mathbf{X}$ is an arbitrary vector, we choose this vector so that its r-th component in the coordinate system u^1, u^2 is equal to unity, while all other components are equal to zero. Then for Y^k_{ij} in this coordinate system we get

$$Y^k_{ij} = \sum_{q=1}^{2} R^k_{qij}\,X^q = R^k_{rij}. \tag{8.9}$$

For the components of the vector $\mathbf{X}$ in the other coordinate system we derive

$$\tilde X^m = \sum_{q=1}^{2} T^m_q\,X^q = T^m_r,$$

then we apply the formula (8.8) on order to calculate the components of the tensor $\mathbf{Y}$ in the second coordinate system:

$$\tilde Y^n_{pq} = \sum_{m=1}^{2} \tilde R^n_{mpq}\,\tilde X^m = \sum_{m=1}^{2} \tilde R^n_{mpq}\,T^m_r. \tag{8.10}$$

The rest is to relate the quantities Y^k_{ij} from (8.9) and the quantities $\tilde Y^n_{pq}$ from (8.10). From the statement of the theorem we know that these quantities are the components of the same tensor in two different coordinate systems. Hence, we get

$$Y^k_{ij} = \sum_{n=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2} S^k_n\,T^p_i\,T^q_j\;\tilde Y^n_{pq}. \tag{8.11}$$

Substituting (8.9) and (8.10) into the formula (8.11), we find

$$R^k_{rij} = \sum_{n=1}^{2}\sum_{m=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2} S^k_n\,T^m_r T^p_i\,T^q_j\;\tilde R^n_{mpq}.$$

This formula exactly coincides with the transformation rule for the components of a tensorial field of the type $(1,3)$ under a change of coordinates. Thus, the lemma is proved. $\square$

The theorem 8.1 is an immediate consequence of the lemma 8.1. Indeed, the left hand side of the formula (8.6) defines a tensor of the type $(1,2)$ for any choice of the vector field $\mathbf{X}$, while the right hand side is the contraction of $\mathbf{R}$ and $\mathbf{X}$.

The components of the curvature tensor given by the formula (8.4) are enumerated by three lower indices and one upper index. Upon lowering by means of the metric tensor the upper index is usually written in the first position:

$$R_{qrij}=\sum_{k=1}^{2}R^{k}_{rij}\,g_{kq}. \tag{8.12}$$

The tensor of the type $(0,4)$ given by the formula (8.12) is denoted by the same letter $\mathbf{R}$. Another tensor is derived from (8.4) by raising the first lower index:

$$R^{kq}_{ij}=\sum_{r=1}^{2}R^{k}_{rij}\,g^{rq}. \tag{8.13}$$

The raised lower index is usually written as the second upper index. The tensors of the type $(0,4)$ and $(2,2)$ with the components (8.12) and (8.13) are denoted by the same letter $\mathbf{R}$ and called the curvature tensors.

THEOREM 8.2. *The components of the curvature tensor* $\mathbf{R}$ *determined by the connection* (7.5) *according to the formula* (8.4) *satisfy the following relationships:*

(1) $R^{k}_{rij}=-R^{k}_{rji}$;

(2) $R_{qrij}=-R_{rqij}$;

(3) $R_{qrij}=R_{ijqr}$;

(4) $R^{k}_{rij}+R^{k}_{ijr}+R^{k}_{jri}=0$.

PROOF. The first relationship is an immediate consequence of the formula (8.4) itself. When transposing the indices i and j the right hand side of (8.4) changes the sign. Hence, we get the identity (1) which means that the curvature tensor is skew-symmetric with respect to the last pair of its lower indices.

In order to prove the identity in the item (2) we apply (8.7) to the metric tensor. As a result we get the following equality:

$$(\nabla_i\nabla_j-\nabla_j\nabla_i)\,g_{qr}=\sum_{k=1}^{2}\bigl(R^{k}_{qij}\,g_{kr}+R^{k}_{rij}\,g_{qk}\bigr).$$

Taking into account (8.12), this equality can be rewritten as

$$(\nabla_i\nabla_j-\nabla_j\nabla_i)\,g_{qr}=R_{rqij}+R_{qrij}. \tag{8.14}$$

Remember that due to the concordance of the metric and connection the covariant derivatives of the metric tensor are equal to zero (see formula (7.1)). Hence, the left hand side of the equality (8.14) is equal to zero, and as a consequence we get the identity from the item (2) of the theorem.

Let's drop for a while the third item of the theorem and prove the fourth item by means of the direct calculations on the base of the formula (8.4). Let's write the relationship (8.4) and perform twice the cyclic transposition of the indices in it: $i \to j \to r \to i$. As a result we get the following three equalities:

$$
\begin{aligned}
R^k_{rij} &= \frac{\partial \Gamma^k_{jr}}{\partial u^i} - \frac{\partial \Gamma^k_{ir}}{\partial u^j} + \sum_{q=1}^{2} \Gamma^k_{iq}\Gamma^q_{jr} - \sum_{q=1}^{2} \Gamma^k_{jq}\Gamma^q_{ir},\\
R^k_{ijr} &= \frac{\partial \Gamma^k_{ri}}{\partial u^j} - \frac{\partial \Gamma^k_{ji}}{\partial u^r} + \sum_{q=1}^{2} \Gamma^k_{jq}\Gamma^q_{ri} - \sum_{q=1}^{2} \Gamma^k_{rq}\Gamma^q_{ji},\\
R^k_{jri} &= \frac{\partial \Gamma^k_{ij}}{\partial u^i} - \frac{\partial \Gamma^k_{rj}}{\partial u^i} + \sum_{q=1}^{2} \Gamma^k_{rq}\Gamma^q_{ij} - \sum_{q=1}^{2} \Gamma^k_{iq}\Gamma^q_{rj}.
\end{aligned}
$$

Let's add all the three above equalities and take into account the symmetry of the Christoffer symbols with respect to their lower indices. It is easy to see that the sum in the right hand side will be zero. This proves the item (4) of the theorem.

The third item of the theorem follows from the first, the second, and the third items. In the left hand side of the equality that we need to prove we have R_{qrij}. The simultaneous transposition of the indices $q \leftrightarrow r$ and $i \leftrightarrow j$ does not change this quantity, i. e. we have the equality

$$R_{qrij} = R_{rqji}. \tag{8.15}$$

This equality follows from the item (1) and the item (2). Let's apply the item (4) to the quantities in both sides of the equality (8.15):

$$
\begin{aligned}
R_{qrij} &= -R_{qijr} - R_{qjri},\\
R_{rqji} &= -R_{rjiq} - R_{riqj}.
\end{aligned} \tag{8.16}
$$

Now let's perform the analogous manipulations with the quantity R_{ijqr}:

$$R_{ijqr} = R_{jirq}. \tag{8.17}$$

To each quantity in (8.17) we apply the item (4) of the theorem:

$$
\begin{aligned}
R_{ijqr} &= -R_{iqrj} - R_{irjq},\\
R_{jirq} &= -R_{jrqi} - R_{jqir}.
\end{aligned} \tag{8.18}
$$

Let's add the equalities (8.16) and subtract from the sum the equalities (8.18). It is easy to verify that due to the items (1) and (2) of the theorem the right hand side of the resulting equality is zero. Then, using (8.15) and (8.17), we get

$$2\,R_{qrij} - 2\,R_{ijqr} = 0.$$

Dividing by 2, now we get the identity that we needed to prove. Thus, the theorem is completely proved. □

The curvature tensor $\mathbf{R}$ given by its components (8.4) has the indices on both levels. Therefore, we can consider the contraction:

$$R_{rj} = \sum_{k=1}^{2} R^k_{rkj}. \tag{8.19}$$

The formula (8.19) for R_{rj} can be transformed as follows:

$$R_{rj} = \sum_{i=1}^{2} \sum_{k=1}^{2} g^{ik}\, R_{irkj}.$$

From this equality due to the symmetry g^{ik} and due to the item (4) of the theorem 8.2 we derive the symmetry of the tensor R_{rj}:

$$R_{rj} = R_{jr}. \tag{8.20}$$

The symmetric tensor of the type $(0, 2)$ with the components (8.19) is called the *Ricci tensor*. It is denoted by the same letter $\mathbf{R}$ as the curvature tensor.

Note that there are other two contractions of the curvature tensor. However, these contractions do not produce new tensors:

$$\sum_{k=1}^{2} R^k_{krj} = 0, \qquad\qquad \sum_{k=1}^{2} R^k_{rik} = -R_{ri}.$$

Using the Ricci tensor, one can construct a scalar field R by means of the formula

$$R = \sum_{r=1}^{2} \sum_{j=1}^{2} R_{rj}\, g^{rj}. \tag{8.21}$$

The scalar $R(u^1, u^2)$ defined by the formula (8.21) is called the *scalar curvature* of a surface at the point with the coordinats u^1, u^2. The scalar curvature is a result of total contraction of the curvature tensor $\mathbf{R}$ given by the formula (8.13):

$$R = \sum_{i=1}^{2} \sum_{j=1}^{2} R^{ij}_{ij}. \tag{8.22}$$

The formula (8.22) is easily derived from (8.21). Any other ways of contracting the curvature tensor do not give other scalars essentially different from (8.21).

In general, passing from the components of the curvature tensor R^{kr}_{ij} to the scalar curvature, we should lose a substantial part of the information contained in the tensor $\mathbf{R}$: this means that we replace 16 quantities by the only one. However, due to the theorem 8.2 in two-dimensional case we do not lose the information at all. Indeed, due to the theorem 8.2 the components of the curvature tensor R^{kr}_{ij} are skew-symmetric both with respect to upper and lower indices. If $k = r$ or $i = j$, they do vanish. Therefore, the only nonzero components are R^{12}_{12}, R^{21}_{12}, R^{12}_{21}, R^{21}_{21}, and they satisfy the equalities $R^{12}_{12} = R^{21}_{21} = -R^{21}_{12} = -R^{12}_{21}$. Hence, we get

$$R = R^{12}_{12} + R^{21}_{21} = 2\, R^{12}_{12}.$$

Let's consider the tensor $\mathbf{D}$ with the following components:

$$D^{kr}_{ij} = \frac{R}{2} \left(\delta^k_i\, \delta^r_j - \delta^k_j\, \delta^r_i\right).$$

The tensor $\mathbf{D}$ is also skew-symmetric with respect to upper and lower indices and $D^{12}_{12} = R^{12}_{12}$. Hence, these tensors do coincide: $\mathbf{D} = \mathbf{R}$. In coordinates we have

$$R^{kr}_{ij} = \frac{R}{2}\left(\delta^k_i\,\delta^r_j - \delta^k_j\,\delta^r_i\right). \tag{8.23}$$

By lowering the upper index r, from (8.23) we derive

$$R^k_{rij} = \frac{R}{2}\left(\delta^k_i\,g_{rj} - \delta^k_j\,g_{ri}\right). \tag{8.24}$$

The formula (8.24) determines the components of the curvature tensor on an arbitrary surface. For the Ricci tensor this formula yields

$$R_{ij} = \frac{R}{2}\,g_{ij}. \tag{8.25}$$

The Ricci tensor of an arbitrary surface is proportional to the metric tensor.

A remark. The curvature tensor determined by the symmetric connection (7.5) possesses another one (fifth) property expressed by the identity

$$\nabla_k R^q_{rij} + \nabla_i R^q_{rjk} + \nabla_j R^q_{rki} = 0. \tag{8.26}$$

The relationship (8.23) is known as the *Bianchi identity*. However, in the case of surfaces (in the dimension 2) it appears to be a trivial consequence from the item (1) of the theorem 8.2. Therefore, we do not give it here.

§ 9. Gauss equation and Peterson-Codazzi equation.

Let's consider the Veingarten's derivational formulas (4.11). They can be treated as a system of nine vectorial equations with respect to three vector-functions $\mathbf{E}_1(u^1,u^2)$, $\mathbf{E}_2(u^1,u^2)$, and $\mathbf{n}(u^1,u^2)$. So, the number of the equations is greater than the number functions. Such systems are said to be *overdetermined*. Overdetermined systems are somewhat superfluous. One usually can derive new equations of the same or lower order from them. Such equations are called *differential consequences* or *compatibility conditions* of the original equations.

As an example we consider the following system of two partial differential equations with respect to the function $f = f(x,y)$:

$$\frac{\partial f}{\partial x} = a(x,y), \qquad\qquad \frac{\partial f}{\partial y} = b(x,y). \tag{9.1}$$

Let's differentiate the first equation (9.1) with respect to y and the second equation with respect to x. Then we subtract one from another:

$$\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x} = 0. \tag{9.2}$$

The equation (9.2) is a compatibility condition for the equations (9.1). It is a necessary condition for the existence of the function satisfying the equations (9.1).

Similarly, one can derive the compatibility conditions for the system of Veingarten's derivational equations (4.11). Let's write the first of them as

$$\frac{\partial \mathbf{E}_k}{\partial u^j} = \sum_{q=1}^{2} \Gamma^q_{jk} \cdot \mathbf{E}_q + b_{jk} \cdot \mathbf{n}. \tag{9.3}$$

Then we differentiate (9.3) with respect to u^i and express the derivatives $\partial \mathbf{E}_k/\partial u^i$ and $\partial \mathbf{n}/\partial u^i$ arising therein by means of the derivational formulas (4.11):

$$\begin{aligned} \frac{\partial \mathbf{E}_k}{\partial u^i\,\partial u^j} &= \left(\frac{\partial b_{jk}}{\partial u^i} + \sum_{q=1}^{2} \Gamma^q_{jk}\, b_{iq}\right) \cdot \mathbf{n} + \\ &+ \sum_{q=1}^{2} \left(\frac{\partial \Gamma^q_{jk}}{\partial u^i} + \sum_{s=1}^{2} \Gamma^s_{jk}\,\Gamma^q_{is} - b_{jk}\, b^q_i\right) \cdot \mathbf{E}_q. \end{aligned} \tag{9.4}$$

Let's transpose indices i and j in the formula (9.4). The value of the second order mixed partial derivative does not depend on the order of differentiation. Therefore, the value of the left hand side of (9.4) does not change under the transposition of indices i and j. Let's subtract from (9.4) the relationship obtained by transposing the indices. As a result we get

$$\begin{gathered} \sum_{q=1}^{2} \left(\frac{\partial \Gamma^q_{jk}}{\partial u^i} - \frac{\partial \Gamma^q_{ik}}{\partial u^j} + \sum_{s=1}^{2} \Gamma^s_{jk}\,\Gamma^q_{is} - \sum_{s=1}^{2} \Gamma^s_{ik}\,\Gamma^q_{js} + b_{ik}\, b^q_j - b_{jk}\, b^q_i\right) \cdot \mathbf{E}_q + \\ + \left(\frac{\partial b_{jk}}{\partial u^i} + \sum_{q=1}^{2} \Gamma^q_{jk}\, b_{iq} - \frac{\partial b_{ik}}{\partial u^j} - \sum_{q=1}^{2} \Gamma^q_{ik}\, b_{jq}\right) \cdot \mathbf{n} = 0. \end{gathered}$$

The vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{n}$ composing the moving frame are linearly independent. Therefore the above equality can be broken into two separate equalities

$$\begin{aligned} &\frac{\partial \Gamma^q_{jk}}{\partial u^i} - \frac{\partial \Gamma^q_{ik}}{\partial u^j} + \sum_{s=1}^{2} \Gamma^s_{jk}\,\Gamma^q_{is} - \sum_{s=1}^{2} \Gamma^s_{ik}\,\Gamma^q_{js} = b_{jk}\, b^q_i - b_{ik}\, b^q_j, \\ &\frac{\partial b_{jk}}{\partial u^i} - \sum_{q=1}^{2} \Gamma^q_{ik}\, b_{jq} = \frac{\partial b_{ik}}{\partial u^j} - \sum_{q=1}^{2} \Gamma^q_{jk}\, b_{iq}. \end{aligned}$$

Note that the left hand side of the first of these relationships coincides with the formula for the components of the curvature tensor (see (8.4)). Therefore, we can rewrite the first relationship as follows:

$$R^q_{kij} = b_{jk}\, b^q_i - b_{ik}\, b^q_j. \tag{9.5}$$

The second relationship can also be simplified:

$$\nabla_i b_{jk} = \nabla_j b_{ik}. \tag{9.6}$$

It is easy to verify this fact immediately by transforming (9.6) back to the initial form applying the formula (6.1).

The equations (9.5) and (9.6) are differential consequences of the Veingarten's derivational formulas (4.11). The first of them is known as the *Gauss equation* and the second one is known as the *Peterson-Codazzi equation*.

The tensorial Gauss equation (9.5) contains 16 separate equalities. However, due to the relationship (8.24) not all of them are independent. In order to simplify (9.5) let's raise the index k in it. As a result we get

$$\frac{R}{2}\left(\delta^q_i\,\delta^k_j-\delta^q_j\,\delta^k_i\right)=b^q_i\,b^k_j-b^q_j\,b^k_i. \tag{9.7}$$

The expression in right hand side of (9.7) is skew-symmetric both with respect to upper and lower pairs of indices and each index in (9.7) runs over only two values. Therefore the right hand side of the equation (9.7) can be transformed as

$$b^q_i\,b^k_j-b^q_j\,b^k_i=B\,\left(\delta^q_i\,\delta^k_j-\delta^q_j\,\delta^k_i\right). \tag{9.8}$$

Substituting $q=1$, $k=2$, $i=1$, $j=2$ into (9.8), for B in (9.8) we get

$$B=b^1_1\,b^2_2-b^1_2\,b^2_1=\det(b^k_i)=K,$$

where K is the Gaussian curvature of a surface (see formula (5.12)). The above considerations show that the Gauss equation (9.5) is equivalent to exactly one scalar equation which is written as follows:

$$R=2\,K. \tag{9.9}$$

This equation relates the scalar and Gaussian curvatures of a surface. It is also called the *Gauss equation*.

CHAPTER V

CURVES ON SURFACES

§ 1. Parametric equations of a curve on a surface.

Let $\mathbf{r}(t)$ be the vectorial-parametric equation of a differentiable curve all points of which lie on some differentiable surface. Suppose that a fragment D containing the points of the curve is charted, i. e. it is equipped with curvilinear coordinates u^1, u^2. This means that there is a bijective mapping $\mathbf{u}\colon\ D \to U$ that maps the points of the curve to some domain $U \subset \mathbb{R}^2$. The curve in the chart U is represented not by three, but by two functions of the parameter t:

$$\begin{cases} u^1 = u^1(t), \\ u^2 = u^2(t) \end{cases} \tag{1.1}$$

(compare with the formulas (1.14) from Chapter IV). The inverse mapping $\mathbf{u}^{-1}$ is represented by the vector-function

$$\mathbf{r} = \mathbf{r}(u^1, u^2). \tag{1.2}$$

It determines the radius-vectors of the points of the surface. Therefore, we have

$$\mathbf{r}(t) = \mathbf{r}(u^1(t), u^2(t)). \tag{1.3}$$

Differentiating (1.3) with respect to t, we obtain the vector $\boldsymbol{\tau}$:

$$\boldsymbol{\tau}(t) = \sum_{i=1}^{2} \dot{u}^i \cdot \mathbf{E}_i(u^1(t), u^2(t)). \tag{1.4}$$

This is the tangent vector of the curve (compare with the formulas (1.15) in Chapter IV). The formula (1.4) shows that the vector $\boldsymbol{\tau}$ lies in the tangent plane of the surface. This is the consequence of the fact that the curve in whole lies on the surface.

Under a change of curvilinear coordinates on the surface the derivatives $\dot{u}^i$ are transformed as the components of a tensor of the type $(1, 0)$. They determine the inner (two-dimensional) representation of the vector τ in the chart. The formula (1.4) is used to pass from inner to outer (tree-dimensional) representation of this vector. **Our main goal** in this chapter is to describe the geometry of curves lying on a surface in terms of its two-dimensional representation in the chart.

The length integral is an important object in the theory of curves, see formula (2.3) in Chapter I. Substituting (1.4) into this formula, we get

$$L = \int\limits_a^b \sqrt{\sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,\dot{u}^i\,\dot{u}^j}\;dt. \tag{1.5}$$

The expression under integration in (1.5) is the length of the vector $\boldsymbol{\tau}$ in its inner representation. If $s = s(t)$ is the natural parameter of the curve, then, denoting $du^i = \dot{u}^i\,dt$, we can write the following formula:

$$ds^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,du^i\,du^j. \tag{1.6}$$

The formula (1.6) approves the title «the first quadratic form» for the metric tensor. Indeed, the square of the length differential ds^2 is a quadratic form of differentials of the coordinate functions in the chart. If $t = s$ is the natural parameter of the curve, then there is the equality

$$\sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,\dot{u}^i\,\dot{u}^j = 1 \tag{1.7}$$

that expresses the fact that the length of the tangent vector $\boldsymbol{\tau}$ of a curve in the natural parametrization is equal to unity (see § 2 in Chapter I).

§ 2. Geodesic and normal curvatures of a curve.

Let $t = s$ be the natural parameter of a parametric curve given by the equations (1.1) in curvilinear coordinates on some surface. Let's differentiate the tangent vector $\boldsymbol{\tau}(s)$ of this curve (1.4) with respect to the parameter s. The derivative of the left hand side of (1.4) is given by the Frenet formulas (3.8) from Chapter I:

$$k\cdot\mathbf{n}_{\,\mathrm{curv}} = \sum_{k=1}^{2}\ddot{u}^k\cdot\mathbf{E}_k + \sum_{i=1}^{2}\sum_{j=1}^{2}\dot{u}^i\cdot\frac{\partial\mathbf{E}_i}{\partial u^j}\cdot\dot{u}^j. \tag{2.1}$$

By $\mathbf{n}_{\,\mathrm{curv}}$ we denote the unit normal vector of the curve in order to distinguish it from the unit normal vector $\mathbf{n}$ of the surface. For to calculate the derivatives $\partial\mathbf{E}_i/\partial u^j$ we apply the Veingarten's derivational formulas (4.11):

$$k\cdot\mathbf{n}_{\,\mathrm{curv}} = \sum_{k=1}^{2}\left(\ddot{u}^k + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma^k_{ji}\,\dot{u}^i\,\dot{u}^j\right)\cdot\mathbf{E}_k + \left(\sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}\,\dot{u}^i\,\dot{u}^j\right)\cdot\mathbf{n}. \tag{2.2}$$

Let's denote by $k_{\,\mathrm{norm}}$ the coefficient of the vector $\mathbf{n}$ in the formula (2.2). This quantity is called the *normal curvature* of a curve:

$$k_{\,\mathrm{norm}} = \sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}\,\dot{u}^i\,\dot{u}^j. \tag{2.3}$$

In contrast to the curvature k, which is always a non-negative quantity, the normal curvature of a curve (2.3) can be either positive, or zero, or negative. Taking into account (2.3), we can rewrite the relationship (2.2) itself as follows:

$$k\cdot\mathbf{n}_{\,\mathrm{curv}} - k_{\,\mathrm{norm}}\cdot\mathbf{n} = \sum_{k=1}^{2}\left(\ddot{u}^k + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma^k_{ji}\,\dot{u}^i\,\dot{u}^j\right)\cdot\mathbf{E}_k. \tag{2.4}$$

The vector in the right hand side of (2.4) is a linear combination of the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ that compose a basis in the tangent plane. Therefore, this vector lies in the tangent plane. Its length is called the *geodesic curvature* of a curve:

$$k_{\,\text{geod}} = \left|\sum_{k=1}^{2}\left(\ddot{u}^k + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma^k_{ji}\,\dot{u}^i\,\dot{u}^j\right)\cdot\mathbf{E}_k\right|. \tag{2.5}$$

Due to the formula (2.5) the geodesic curvature of a curve is always non-negative. If $k_{\,\text{geod}} \neq 0$, then, taking into account the relationship (2.5), one can define the unit vector $\mathbf{n}_{\,\text{inner}}$ and rewrite the formula (2.4) as follows:

$$k\cdot\mathbf{n}_{\,\text{curv}} - k_{\,\text{norm}}\cdot\mathbf{n} = k_{\,\text{geod}}\cdot\mathbf{n}_{\,\text{inner}}. \tag{2.6}$$

The unit vector $\mathbf{n}_{\,\text{inner}}$ in the formula (2.6) is called the *inner normal vector* of a curve on a surface.

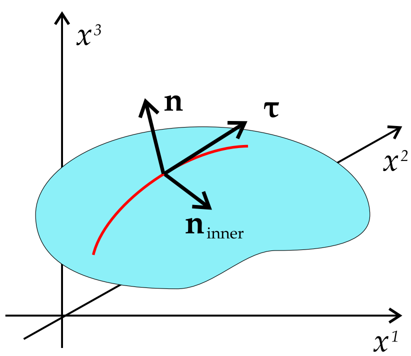

Fig. 2.1

Due to (2.6) the vector $\mathbf{n}_{\,\text{inner}}$ is a linear combination of the vectors $\mathbf{n}_{\,\text{curv}}$ and $\mathbf{n}$ which are perpendicular to the unit vector $\boldsymbol{\tau}$ lying in the tangent plane. Hence, $\mathbf{n}_{\,\text{inner}} \perp \boldsymbol{\tau}$. On the other hand, being a linear combination of the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, the vector $\mathbf{n}_{\,\text{inner}}$ itself lies in the tangent plane. Therefore, it is determined up to the sign:

$$\mathbf{n}_{\,\text{inner}} = \pm[\boldsymbol{\tau},\,\mathbf{n}]. \tag{2.7}$$

Using the relationship (2.7), sometimes one can extend the definition of the vector $\mathbf{n}_{\,\text{inner}}$ even to those points of the curve where $k_{\,\text{geod}} = 0$.

Let's move the term $k_{\,\text{norm}}\cdot\mathbf{n}$ to the right hand side of the formula (2.6). Then this formula is rewritten as follows:

$$k\cdot\mathbf{n}_{\,\text{curv}} = k_{\,\text{geod}}\cdot\mathbf{n}_{\,\text{inner}} + k_{\,\text{norm}}\cdot\mathbf{n}. \tag{2.8}$$

The relationship (2.8) can be treated as an expansion of the vector $k\cdot\mathbf{n}_{\,\text{curv}}$ as a sum of two mutually prpendicular components. Hence, we have

$$k^2 = (k_{\,\text{geod}})^2 + (k_{\,\text{norm}})^2. \tag{2.9}$$

The formula (2.3) determines the value of the normal curvature of a curve in the natural parametrization $t = s$. Let's rewrite it as follows:

$$k_{\,\text{norm}} = \frac{\displaystyle\sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}\,\dot{u}^i\,\dot{u}^j}{\displaystyle\sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,\dot{u}^i\,\dot{u}^j}. \tag{2.10}$$

In the natural parametrization the formula (2.10) coincides with (2.3) because of (1.7). When passing to an arbitrary parametrization all derivatives $\dot{u}^i$ are multiplied by the same factor. Indeed, we have

$$\frac{du^i}{dt} = \frac{du^i}{ds}\,\frac{ds}{dt}. \tag{2.11}$$

But the right hand side of (2.10) is insensitive to such a change of $\dot{u}^i$. Therefore, (2.10) is a valid formula for the normal curvature in any parametrization.

The formula (2.10) shows that the normal curvature is a very rough characteristic of a curve. It is determined only by the direction of its tangent vector $\boldsymbol{\tau}$ in the tangent plane. The components of the matrices g_{ij} and b_{ij} characterize not the curve, but the point of the surface through which this curve passes.

Let $\mathbf{a}$ be some vector tangent to the surface. In curvilinear coordinates u^1, u^2 it is given by two numbers a^1, a^2 — they are the coefficients in its expansion in the basis of two frame vectors $\mathbf{E}_1$ and $\mathbf{E}_2$. Let's consider the value of the second quadratic form of the surface on this vector:

$$\mathbf{b}(\mathbf{a},\mathbf{a}) = \sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}\,a^i\,a^j. \tag{2.12}$$

Definition 2.1. The direction given by a nonzero vector $\mathbf{a}$ in the tangent plane is called an *asymptotic direction* if the value of the second quadratic form (2.12) on such vector is equal to zero.

Note that asymptotic directions do exist only at those points of a surface where the second quadratic form is indefinite in sign or degenerate. In the first case the Gaussian curvature is negative: $K < 0$, in the second case it is equal to zero: $K = 0$. At those points where $K > 0$ there are no asymptotic directions.

Definition 2.2. A curve on a surface whose tangent vector $\boldsymbol{\tau}$ at all its points lies in an asymptotic direction is called an *asymptotic line*.

Due to (2.12) the equation of an asymptotic line has the following form:

$$\mathbf{b}(\boldsymbol{\tau},\boldsymbol{\tau}) = \sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}\,\dot{u}^i\,\dot{u}^j = 0. \tag{2.13}$$

Comparing (2.13) and (2.10), we see that asymptotic lines are the lines with zero normal curvature: $k_{\,\text{norm}} = 0$. On the surfaces with negative Gaussian curvature $K < 0$ at each point there are two asymptotic directions. Therefore, on such surfaces always there are two families of asymptotic lines, they compose the *asymptotic network* of such a surface. On any surface of the negative Gaussian curvature there exists a curvilinear coordinate system u^1, u^2 whose coordinate network coincides with the asymptotic network of this surface. However, we shall not prove this fact here.

The *curvature lines* are defined by analogy with the asymptotic lines. These are the curves on a surface whose tangent vector lies in a principal direction at each point (see formulas (5.14) and (5.15) in § 5 of Chapter IV). The curvature lines do exist on any surface, there are no restrictions for the Gaussian curvature of a surface in this case.

Definition 2.3. A *geodesic line* on a surface is a curve whose geodesic curvature is identically equal to zero: $k_{\,\text{geod}} = 0$.

From the Frenet formula $\dot{\boldsymbol{\tau}} = k \cdot \mathbf{n}_{\,\text{curv}}$ and from the relationship (2.8) for the geodesic lines we derive the following equality:

$$\frac{d\boldsymbol{\tau}}{ds} = k_{\,\text{norm}} \cdot \mathbf{n}. \tag{2.14}$$

In the other words, the derivative of the unit normal vector on a geodesic line is directed along the unit normal vector of a surface. This is the external description of geodesic lines. The inner description is derived from the formula (2.5):

$$\ddot{u}^k + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma^k_{ji}\,\dot{u}^i\,\dot{u}^j = 0. \tag{2.15}$$

The equations (2.15) are the *differential equations of geodesic lines in natural parametrization*. One can pass from the natural parametrization to an arbitrary one by means of the formula (2.11).

§ 3. Extremal property of geodesic lines.

Let's compare the equations of geodesic lines (2.15) with the equations of straight lines in curvilinear coordinates (8.18) which we have derived in Chapter III. These equations have the similar structure. They differ only in the ranges of indices: in the case of geodesic lines on a surface they run over two numbers instead of three numbers in the case of straight lines. Therefore, geodesic lines are natural analogs of the straight lines in the inner geometry of surfaces. The following theorem strengthens this analogy.

Theorem 3.1. *A geodesic line connecting two given points A and B on a surface has the extremal length in the class of curves connecting these two points.*

It is known that in the Euclidean space $\mathbb{E}$ the shortest path from a point A to another point B is the segment of straight line connecting these points. The theorem 3.1 proclaims a very similar proposition for geodesic lines on a surface. Remember that real functions can have local maxima and minima — they are called *extrema*. Apart from maxima and minima, there are also *conditional extrema* (saddle points), for example, the point $x = 0$ for the function $y = x^3$. All those points are united by the common property — the linear part of the function increment at any such point is equal to zero: $f(x_0 + h) = f(x_0) + O(h^2)$.

In the case of a geodesic line connecting the points A and B on a surface we should slightly deform (variate) this line keeping it to be a line on the surface connecting the same two points A and B. The deformed line is not a geodesic line. Its length differs from the length of the original line. The condition of the *extremal length* in the theorem 3.1 means that the linear part of the length increment is equal to zero.

Let's specify the method of deforming the curve. For the sake of simplicity assume that the points A and B, and the geodesic line connecting these points in whole lie within some charted fragment D of the surface. Then this geodesic line is given by two functions (1.1). Let's increment by one the number of arguments in these functions. Then we shall assume that these functions are sufficiently many times differentiable with respect to all their arguments:

$$\begin{cases} u^1 = u^1(t, h), \\ u^2 = u^2(t, h). \end{cases} \tag{3.1}$$

For each fixed h in (3.1) we have the functions of the parameter t, they define a curve on the surface. Changing the parameter h, we deform the curve so that in the process of this deformation its point are always on the surface. The differentiability of the functions (3.1) guarantees that small deformations of the curve correspond to small changes of the parameter h.

Let's impose to the functions (3.1) a series of restrictions which are easy to satisfy. Assume that the length of the initial geodesic line is equal to a and let the parameter t run over the segment $[0, a]$. Let

$$u^k(0, h) = u^k(0, 0), \qquad\qquad u^k(a, h) = u^k(a, 0). \tag{3.2}$$

The condition (3.2) means that under a change of the parameter h the initial point A and the ending point B of the curve do not move.

For the sake of brevity let's denote the partial derivatives of the functions $u^i(t, h)$ with respect to t by setting the dot. Then the quantities $\dot{u}^i = \partial u^i/\partial t$ determine the inner representation of the tangent vector to the curve.

Assume that the initial line correspond to the value $h = 0$ of the parameter h. Assume also that for $h = 0$ the parameter t coincides with the natural parameter of the geodesic line. Then for $h = 0$ the functions (3.1) satisfy the equations (1.7) and (2.15) simultaneously. For $h \neq 0$ the parameter t should not coincide with the natural parameter on the deformed curve, and the deformed curve itself should not be a geodesic line in this case.

Let's calculate the lengths of the deformed curves. It is the function of the parameter h determined by the length integral of the form (1.5):

$$L(h) = \int\limits_0^a \sqrt{\sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,\dot{u}^i\,\dot{u}^j}\;dt. \tag{3.3}$$

For $h = 0$ we have $L(0) = a$. The proposition of the theorem 3.1 on the extremity of the length now is formulated as $L(h) = a + O(h^2)$ or, equivalently, as

$$\left.\frac{dL(h)}{dh}\right|_{h=0} = 0. \tag{3.4}$$

Proof of the theorem 3.1. Let's prove the equality (3.4) for the length integral (3.3) under the deformations of the curve described just above. Denote by $\lambda(t, h)$ the expression under the square root in the formula (3.3). Then by direct differentiation of (3.3) we obtain

$$\frac{dL(h)}{dh} = \int\limits_0^a \frac{\partial\lambda/\partial h}{2\sqrt{\lambda}}\,dt. \tag{3.5}$$

Let's calculate the derivative in the numerator of the fraction (3.5):

$$\frac{\partial\lambda}{\partial h}=\frac{\partial}{\partial h}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}g_{ij}\,\dot u^i\,\dot u^j\right)=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial g_{ij}}{\partial u^k}\frac{\partial u^k}{\partial h}\,\dot u^i\,\dot u^j+$$

$$+\sum_{k=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2}g_{ij}\,\frac{\partial(\dot u^i\,\dot u^j)}{\partial\dot u^k}\,\frac{\partial\dot u^k}{\partial h}=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial g_{ij}}{\partial u^k}\frac{\partial u^k}{\partial h}\,\dot u^i\,\dot u^j+$$

$$+\sum_{k=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2}g_{ij}\,\delta^i_k\,\dot u^j\,\frac{\partial\dot u^k}{\partial h}+\sum_{k=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2}g_{ij}\,\dot u^i\,\delta^j_k\,\frac{\partial\dot u^k}{\partial h}.$$

Due to the Kronecker symbols δ^i_k and δ^j_k in the above expression we can perform explicitly the summation over k in the last two terms. Moreover, due to the symmetry of g_{ij} they are equal to each other:

$$\frac{\partial\lambda}{\partial h}=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial g_{ij}}{\partial u^k}\frac{\partial u^k}{\partial h}\,\dot u^i\,\dot u^j+2\sum_{i=1}^{2}\sum_{j=1}^{2}g_{ij}\,\dot u^i\,\frac{\partial\dot u^j}{\partial h}.$$

Substituting this expression into (3.5), we get two integrals:

$$I_1=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\int\limits_0^a\frac{\partial g_{ij}}{\partial u^k}\,\frac{\dot u^i\,\dot u^j}{2\sqrt{\lambda}}\,\frac{\partial u^k}{\partial h}\,dt,\tag{3.6}$$

$$I_2=\sum_{i=1}^{2}\sum_{j=1}^{2}\int\limits_0^a\frac{g_{ik}\,\dot u^i}{\sqrt{\lambda}}\,\frac{\partial\dot u^k}{\partial h}\,dt.\tag{3.7}$$

The integral (3.7) contain the second order mixed partial derivatives of (3.1):

$$\frac{\partial\dot u^k}{\partial h}=\frac{\partial^2u^k}{\partial t\,\partial h}.$$

In order to exclude such derivatives we integrate (3.7) by parts:

$$\int\limits_0^a\frac{g_{ik}\,\dot u^i}{\sqrt{\lambda}}\,\frac{\partial\dot u^k}{\partial h}\,dt=\frac{g_{ik}\,\dot u^i}{\sqrt{\lambda}}\,\frac{\partial u^k}{\partial h}\bigg|_0^a-\int\limits_0^a\frac{\partial}{\partial t}\left(\frac{g_{ik}\,\dot u^i}{\sqrt{\lambda}}\right)\frac{\partial u^k}{\partial h}\,dt.$$

Let's differentiate the equalities (3.2) with respect to h. As a result we find that the derivatives $\partial u^k/\partial h$ vanish at the ends of the integration segment over t. This means that non-integral terms in the above formula do vanish. Hence, for the integral I_2 in (3.7) we obtain

$$I_2=-\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_0^a\frac{\partial}{\partial t}\left(\frac{g_{ik}\,\dot u^i}{\sqrt{\lambda}}\right)\frac{\partial u^k}{\partial h}\,dt.\tag{3.8}$$

Now let's add the integrals I_1 and I_2 from (3.6) and (3.8). As a result for the derivative dL/dh in (3.5) we derive the following equality:

$$\frac{dL(h)}{dh}=\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_{0}^{a}\left(\sum_{j=1}^{2}\frac{\partial g_{ij}}{\partial u^k}\,\frac{\dot u^i\,\dot u^j}{2\sqrt{\lambda}}-\frac{\partial}{\partial t}\left(\frac{g_{ik}\,\dot u^i}{\sqrt{\lambda}}\right)\right)\frac{\partial u^k}{\partial h}\,dt.$$

In this equality the only derivatives with respect to the parameter h are $\partial u^k/\partial h$. For their values at $h=0$ we introduce the following notations:

$$\delta u^k=\frac{\partial u^k}{\partial h}\bigg|_{h=0}\tag{3.9}$$

The quantities $\delta u^k=\delta u^k(t)$ in (3.9) are called the *variations of the coordinates* on the initial curve. Note that under a change of curvilinear coordinates these quantities are transformed as the components of a vector (although this fact does not matter for proving the theorem).

Let's substitute $h=0$ into the above formula for the derivative dL/dh. When substituted, the quantity λ in the denominators of the fractions becomes equal to unity: $\lambda(t,0)=1$. This fact follows from (1.7) since t coincides with the natural parameter on the initial geodesic line. Then

$$\frac{dL(h)}{dh}\bigg|_{h=0}=\sum_{i=1}^{2}\sum_{k=1}^{2}\int\limits_{0}^{a}\left(\sum_{j=1}^{2}\frac{\partial g_{ij}}{\partial u^k}\,\frac{\dot u^i\,\dot u^j}{2}-\frac{d(g_{ik}\,\dot u^i)}{dt}\right)\delta u^k\,dt.$$

Since the above equality does not depend on h any more, we replace the partial derivative with respect to t by d/dt. All of the further calculations in the right hand side are for the geodesic line where t is the natural parameter.

Let's move the sums over i and k under the integration and let's calculate the coefficients of δu^k denoting these coefficients by U_k:

$$\begin{aligned}U_k&=\sum_{i=1}^{2}\left(\sum_{j=1}^{2}\frac{\partial g_{ij}}{\partial u^k}\,\frac{\dot u^i\,\dot u^j}{2}-\frac{d(g_{ik}\,\dot u^i)}{dt}\right)=\\&=\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\frac{1}{2}\frac{\partial g_{ij}}{\partial u^k}-\frac{\partial g_{ik}}{\partial u^j}\right)\dot u^i\,\dot u^j-\sum_{i=1}^{2}g_{ik}\,\ddot u^i.\end{aligned}\tag{3.10}$$

Due to the symmetry of $\dot u^i\,\dot u^j$ the second term within round brackets in the formula (3.10) can be broken into two terms. This yields

$$U_k=\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial u^k}-\frac{\partial g_{ik}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^i}\right)\dot u^i\,\dot u^j-\sum_{i=1}^{2}g_{ik}\,\ddot u^i.$$

Let's raise the index k in U_k, i. e. consider the quantities U^q given by the formula

$$U^q=\sum_{k=1}^{2}g^{qk}\,U_k.$$

For these quantities from the previously derived formula we obtain

$$-U^q = \ddot{u}^q + \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{g^{qk}}{2}\left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right)\dot{u}^i\,\dot{u}^j.$$

Let's compare this formula with the formula (7.5) in Chapter IV that determines the connection components. As a result we get:

$$-U^q = \ddot{u}^q + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma^q_{ij}\,\dot{u}^i\,\dot{u}^j. \tag{3.11}$$

Now it is sufficient to compare (3.11) with the equation of geodesic lines (2.15) and derive $U^q = 0$. The quantities U_k are obtained from U^q by lowering the index:

$$U_k = \sum_{q=1}^{2} g_{kq}\,U^q.$$

Therefore, the quantities U_k are also equal to zero. From this fact we immediately derive the equality (3.4) which means exactly that the extremity condition for the geodesic lines is fulfilled. The theorem is proved. □

§ 4. Inner parallel translation on a surface.

The equation of geodesic lines in the Euclidean space $\mathbb{E}$ in form of (8.18) was derived in Chapter III when considering the parallel translation of vectors in curvilinear coordinates. The differential equation of the parallel translation (8.6) can be rewritten now in the two-dimensional case:

$$\dot{a}^i + \sum_{j=1}^{2}\sum_{k=1}^{2}\Gamma^i_{jk}\,\dot{u}^j\,a^k = 0. \tag{4.1}$$

The equation (4.1) is called the *equation of the inner parallel translation* of vectors along curves on a surface.

Suppose that we have a surface on some fragment of which the curvilinear coordinates u^1, u^2 and a parametric curve (1.1) are given. Let's consider some tangent vector $\mathbf{a}$ to the surface at the initial point of the curve, i.e. at $t = 0$. The vector $\mathbf{a}$ has the inner representation in form of two numbers a^1, a^2, they are its components. Let's set the Cauchy problem for the differential equations (4.1) given by the following initial data at $t = 0$:

$$a^1(t)\Bigr|_{t=0} = a^1, \qquad\qquad a^2(t)\Bigr|_{t=0} = a^2. \tag{4.2}$$

Solving the Cauchy problem (4.2), we get two functions $a^1(t)$ and $a^2(t)$ which determine the vectors $\mathbf{a}(t)$ at all points of the curve. The procedure described just above is called the *inner parallel translation of the vector* $\mathbf{a}$ *along a curve on a surface.*

Let's consider the inner parallel translation of the vector $\mathbf{a}$ from the outer point of view, i. e. as a process in outer (three-dimensional) geometry of the space $\mathbb{E}$ where the surface under consideration is embedded. The relation of inner and outer representations of tangent vectors of the surface is given by the formula:

$$\mathbf{a} = \sum_{i=1}^{2} a^i \cdot \mathbf{E}_i. \tag{4.3}$$

Let's differentiate the equality (4.3) with respect to t assuming that a^1 and a^2 depend on t as solutions of the differential equations (4.1):

$$\frac{d\mathbf{a}}{dt} = \sum_{i=1}^{2} \dot{a}^i \cdot \mathbf{E}_i + \sum_{i=1}^{2}\sum_{j=1}^{2} a^i \cdot \frac{\partial \mathbf{E}_i}{\partial u^j} \cdot \dot{u}^j. \tag{4.4}$$

The derivatives $\partial\mathbf{E}_i/\partial u^j$ are calculated according to Veingarten's derivational formulas (see formulas (4.11) in Chapter IV). Then

$$\frac{d\mathbf{a}}{dt} = \sum_{i=1}^{2}\left(\dot{a}^i + \sum_{j=1}^{2}\sum_{k=1}^{2} \Gamma^i_{jk}\,\dot{u}^j\,a^k\right)\cdot \mathbf{E}_i + \left(\sum_{j=1}^{2}\sum_{k=1}^{2} b_{jk}\,\dot{u}^j\,a^k\right)\cdot \mathbf{n}.$$

Since the functions $a^i(t)$ satisfy the differential equations (4.1), the coefficients at the vectors $\mathbf{E}_i$ in this formula do vanish:

$$\frac{d\mathbf{a}}{dt} = \left(\sum_{j=1}^{2}\sum_{k=1}^{2} b_{jk}\,\dot{u}^j\,a^k\right)\cdot \mathbf{n}. \tag{4.5}$$

The coefficient at the normal vector $\mathbf{n}$ in the above formula (4.5) is determined by the second quadratic form of the surface. This is the value of the corresponding symmetric bilinear form on the pair of vectors $\mathbf{a}$ and $\boldsymbol{\tau}$. Therefore, the formula (4.5) is rewritten in a vectorial form as follows:

$$\frac{d\mathbf{a}}{dt} = \mathbf{b}(\boldsymbol{\tau}, \mathbf{a}) \cdot \mathbf{n}. \tag{4.6}$$

The vectorial equation (4.6) is called the *outer equation of the inner parallel translation on surfaces*.

The operation of parallel translation can be generalized to the case of inner tensors of the arbitrary type (r, s). For this purpose we have introduced the operation of covariant differentiation of tensorial function with respect to the parameter t on curves (see formula (8.10) in Chapter III). Here is the two-dimensional version of this formula:

$$\begin{gathered}\nabla_t A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{dA^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{dt} + \\ + \sum_{m=1}^{r}\sum_{q=1}^{2}\sum_{v_m=1}^{2} \Gamma^{i_m}_{q\,v_m}\,\dot{u}^q\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s} - \sum_{n=1}^{s}\sum_{q=1}^{2}\sum_{w_n=1}^{2} \Gamma^{w_n}_{q\,j_n}\,\dot{u}^q\,A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}.\end{gathered} \tag{4.7}$$

In terms of the covariant derivative (4.7) the equation of the inner parallel translation for the tensorial field $\mathbf{A}$ is written as

$$\nabla_t \mathbf{A} = 0. \tag{4.8}$$

The consistence of defining the inner parallel translation by means of the equation (4.8) follows from the two-dimensional analog of the theorem 8.2 from Chapter III.

Theorem 4.1. *For any inner tensorial function $\mathbf{A}(t)$ determined at the points of a parametric curve on some surface the quantities $B^{i_1 \dots i_r}_{j_1 \dots j_s} = \nabla_t A^{i_1 \dots i_r}_{j_1 \dots j_s}$ calculated according to the formula (4.7) define a tensorial function $\mathbf{B}(t) = \nabla_t \mathbf{A}$ of the same type (r, s) as the original function $\mathbf{A}(t)$.*

The proof of this theorem almost literally coincides with the proof of the theorem 8.2 in Chapter III. Therefore, we do not give it here.

The covariant differentiation ∇_t defined by the formula (4.7) possesses a series of properties similar to those of the covariant differentiation along a vector field $\nabla_{\mathbf{X}}$ (see formula (6.10) and theorem 6.2 in Chapter IV).

Theorem 4.2. *The operation of covariant differentiation of tensor-valued functions with respect to the parameter t along a curve on a surface possesses the following properties:*

(1) $\nabla_t(\mathbf{A} + \mathbf{B}) = \nabla_t \mathbf{A} + \nabla_t \mathbf{B}$;
(2) $\nabla_t(\mathbf{A} \otimes \mathbf{B}) = \nabla_t \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \nabla_t \mathbf{B}$;
(3) $\nabla_t C(\mathbf{A}) = C(\nabla_t \mathbf{A})$.

Proof. Let's choose some curvilinear coordinate system and prove the theorem by means of direct calculations in coordinates. Let $\mathbf{C} = \mathbf{A} + \mathbf{B}$. Then for the components of the tensor-valued function $\mathbf{C}(t)$ we have

$$C^{i_1 \dots i_r}_{j_1 \dots j_s} = A^{i_1 \dots i_r}_{j_1 \dots j_s} + B^{i_1 \dots i_r}_{j_1 \dots j_s}.$$

Substituting $C^{i_1 \dots i_r}_{j_1 \dots j_s}$ into (4.7), for the covariant derivative $\nabla_t \mathbf{C}$ we get

$$\nabla_t C^{i_1 \dots i_r}_{j_1 \dots j_s} = \nabla_t A^{i_1 \dots i_r}_{j_1 \dots j_s} + \nabla_t B^{i_1 \dots i_r}_{j_1 \dots j_s}.$$

This equality proves the first item of the theorem.

Let's proceed with the item (2). Denote $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$. Then for the components of the tensor-valued function $\mathbf{C}(t)$ we have

$$C^{i_1 \dots i_{r+p}}_{j_1 \dots j_{s+q}} = A^{i_1 \dots i_r}_{j_1 \dots j_s} B^{i_{r+1} \dots i_{r+p}}_{j_{s+1} \dots j_{s+q}}. \tag{4.9}$$

Let's substitute the quantities $C^{i_1 \dots i_{r+p}}_{j_1 \dots j_{s+q}}$ from (4.9) into the formula (4.8) for the covariant derivative. As a result for the components of $\nabla_t \mathbf{C}$ we derive

$$\nabla_t C^{i_1 \dots i_{r+p}}_{j_1 \dots j_{s+q}} = \left(dA^{i_1 \dots i_r}_{j_1 \dots j_s}/dt\right) B^{i_{r+1} \dots i_{r+p}}_{j_{s+1} \dots j_{s+q}} +$$

$$+ A^{i_1 \dots i_r}_{j_1 \dots j_s} \left(dB^{i_{r+1} \dots i_{r+p}}_{j_{s+1} \dots j_{s+q}}/dt\right) +$$

$$
\begin{aligned}
&+\sum_{m=1}^{r}\sum_{q=1}^{2}\sum_{v_m=1}^{2}\Gamma^{i_m}_{v_m}\,\dot{u}^q\,A^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}+\\
&+\sum_{m=r+1}^{r+p}\sum_{q=1}^{2}\sum_{v_m=1}^{2}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\Gamma^{i_m}_{v_m}\,\dot{u}^q\,B^{i_{r+1}\ldots\,v_m\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}-\\
&-\sum_{n=1}^{s}\sum_{q=1}^{2}\sum_{w_n=1}^{2}\Gamma^{w_n}_{j_n}\,\dot{u}^q\,A^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}-\\
&-\sum_{n=s+1}^{s+q}\sum_{q=1}^{2}\sum_{w_n=1}^{2}A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\Gamma^{w_n}_{j_n}\,\dot{u}^q\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,w_n\ldots\,j_{s+q}}.
\end{aligned}
$$

Note that upon collecting the similar terms the above huge formula can be transformed to the following one:

$$
\begin{aligned}
&\nabla_t\bigl(A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}\bigr)=\bigl(\nabla_t A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\bigr)\times\\
&\quad\times B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}+A^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\bigl(\nabla_t B^{i_{r+1}\ldots\,i_{r+p}}_{j_{s+1}\ldots\,j_{s+q}}\bigr).
\end{aligned}
\tag{4.10}
$$

Now it is easy to see that the formula (4.10) proves the second item of the theorem.

Let's choose two tensor-valued functions $\mathbf{A}(t)$ and $\mathbf{B}(t)$ one of which is the contraction of another. In coordinates this fact looks like

$$
B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{k=1}^{2}A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}.
\tag{4.11}
$$

Let's substitute (4.11) into the formula (4.7). For $\nabla_t B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ we derive

$$
\begin{aligned}
\nabla_t B^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\sum_{k=1}^{2}\frac{dA^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}}{dt}+\\
&+\sum_{m=1}^{r}\sum_{k=1}^{2}\sum_{q=1}^{2}\sum_{v_m=1}^{2}\Gamma^{i_m}_{q\,v_m}\,\dot{u}^q\,A^{i_1\ldots\,v_m\ldots\,k\,\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s}-\\
&-\sum_{n=1}^{s}\sum_{k=1}^{2}\sum_{q=1}^{2}\sum_{w_n=1}^{2}\Gamma^{w_n}_{q\,j_n}\,\dot{u}^q\,A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,k\,\ldots\,j_s}.
\end{aligned}
\tag{4.12}
$$

In the formula (4.12) the index v_m sequentially occupies the positions to the left of the index k and to the right of it. The same is true for the index w_n. However, the formula (4.12) has no terms where v_m or w_n replaces the index k. Such terms, provided they would be present, according to (4.7), would have the form

$$
\sum_{k=1}^{2}\sum_{q=1}^{2}\sum_{v=1}^{2}\Gamma^{k}_{q\,v}\,\dot{u}^q\;A^{i_1\ldots\,i_{p-1}\,v\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,k\,j_q\ldots\,j_s},
\tag{4.13}
$$

$$
-\sum_{k=1}^{2}\sum_{q=1}^{2}\sum_{w=1}^{2}\Gamma^{w}_{q\,k}\,\dot{u}^q\;A^{i_1\ldots\,i_{p-1}\,k\,i_p\ldots\,i_r}_{j_1\ldots\,j_{q-1}\,w\,j_q\ldots\,j_s}.
\tag{4.14}
$$

It is easy to note that (4.13) and (4.14) differ only in sign. Indeed, it is sufficient to rename k to v and w to k in the formula (4.14). If we add simultaneously (4.13) and (4.14) to (4.12), their contributions cancel each other thus keeping the equality valid. Therefore, (4.12) can be written as

$$\nabla_{j_{s+1}} B^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum_{k=1}^{2} \nabla_{j_{s+1}} A^{i_1 \ldots\, i_{p-1}\, k\, i_p \ldots\, i_r}_{j_1 \ldots\, j_{q-1}\, k\, j_q \ldots\, j_s}. \tag{4.15}$$

The relationship (4.15) proves the third item of the theorem and completes the proof in whole. $\square$

Under a reparametrization of a curve a new parameter $\tilde{t}$ should be a strictly monotonic function of the old parameter t (see details in § 2 of Chapter I). Under such a reparametrization $\nabla_{\tilde{t}}$ and ∇_t are related to each other by the formula

$$\nabla_t \mathbf{A} = \frac{d\tilde{t}(t)}{dt} \cdot \nabla_{\tilde{t}} \mathbf{A} \tag{4.16}$$

for any tensor-valued function $\mathbf{A}$ on a curve. This relationship is a simple consequence from (4.7) and from the chain rule for differentiating a composite function. It is an analog of the item (3) in the theorem 6.2 of Chapter IV.

Let $\mathbf{A}$ be a tensor field of the type (r, s) on a surface. This means that at each point of the surface some tensor of the type (r, s) is given. If we mark only those points of the surface which belong to some curve, we get a tensor-valued function $\mathbf{A}(t)$ on that curve. In coordinates this is written as

$$A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}(t) = A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}(u^1(t), u^2(t)). \tag{4.17}$$

The function $\mathbf{A}(t)$ constructed in this way is called the *restriction of a tensor field* $\mathbf{A}$ to a curve. The specific feature of the restrictions of tensor fields on curves expressed by the formula (4.17) reveals in differentiating them:

$$\frac{dA^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{dt} = \sum_{q=1}^{2} \frac{\partial A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial u^q}\, \dot{u}^q. \tag{4.18}$$

Substituting (4.18) into the formula (4.7), we can extract the common factor $\dot{u}^q$ in the sum over q. Upon extracting this common factor we find

$$\nabla_t A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum_{q=1}^{2} \dot{u}^q\, \nabla_q A^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}. \tag{4.19}$$

The formula (4.19) means that the covariant derivative of the restriction of a tensor field $\mathbf{A}$ to a curve is the contraction of the covariant differential $\nabla\mathbf{A}$ with the tangent vector of the curve.

Assume that $\nabla\mathbf{A} = 0$. Then due to (4.19) the restriction of the field $\mathbf{A}$ to any curve is a tensor-valued function satisfying the equation of the parallel translation (4.8). The values of such a field $\mathbf{A}$ at various points are related to each other by parallel translation along any curve connecting these points.

Definition 4.1. A tensor field $\mathbf{A}$ is called an *autoparallel* field or a *covariantly constant* field if its covariant differential is equal to zero identically: $\nabla\mathbf{A} = 0$.

Some of the well-known tensor fields have identically zero covariant differentials: this is the metric tensor $\mathbf{g}$, the inverse metric tensor $\hat{\mathbf{g}}$, and the area tensor (pseudotensor) $\boldsymbol{\tau}$. The autoparallelism of these fields plays the important role for describing the inner parallel translation.

Let $\mathbf{a}$ and $\mathbf{b}$ be two tangent vectors of the surface at the initial point of some curve. Their scalar product is calculated through their components:

$$(\mathbf{a}\,|\,\mathbf{b}) = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,a^i\,b^j.$$

Let's perform the parallel translation of the vectors $\mathbf{a}$ and $\mathbf{b}$ along the curve solving the equation (4.8) and using the components of $\mathbf{a}$ and $\mathbf{b}$ as initial data in Cauchy problems. As a result we get two vector-valued functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ on the curve. Let's consider the function $\psi(t)$ equal to their scalar product:

$$\psi(t) = (\mathbf{a}(t)\,|\,\mathbf{b}(t)) = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(t)\;a^i(t)\,b^j(t). \tag{4.20}$$

According to the formula (4.7) the covariant derivative $\nabla_t\psi$ coincides with the regular derivative. Therefore, we have

$$\frac{d\psi}{dt} = \nabla_t\psi = \sum_{i=1}^{2}\sum_{j=1}^{2}\bigl(\nabla_t g_{ij}\;a^i\,b^j + g_{ij}\,\nabla_t a^i\;b^j + g_{ij}\,a^i\;\nabla_t b^j\bigr).$$

Here we used the items (2) and (3) of the theorem 4.2. But $\nabla_t a^i = 0$ and $\nabla_t b^j = 0$ since we $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are obtained as a result of parallel translation of the vectors $\mathbf{a}$ and $\mathbf{b}$. Moreover, $\nabla_t g_{ij} = 0$ due to autoparallelism of the metric tensor. For the scalar function $\psi(t)$ defined by (4.20) this yields $d\psi/dt = 0$ and $\psi(t) = (\mathbf{a}\,|\,\mathbf{b}) = \text{const}$. As a result of these considerations we have proved the following theorem.

Theorem 4.3. *The operation of inner parallel translation of vectors along curves preserves the scalar product of vectors.*

Preserving the scalar product, the operation of inner parallel translation preserves the length of vectors and the angles between them.

From the autoparallelism of metric tensors $\mathbf{g}$ and $\hat{\mathbf{g}}$ we derive the following formulas analogous to the formulas (7.9) in Chapter IV:

$$\begin{aligned}
&\nabla_t\left(\sum_{k=1}^{2} g_{ik}\,A^{\dots\,k\,\dots}_{\dots\;\;\dots}\right) = \sum_{k=1}^{2} g_{ik}\,\nabla_t A^{\dots\,k\,\dots}_{\dots\;\;\dots},\\
&\nabla_t\left(\sum_{k=1}^{2} g^{ik}\,A^{\dots\;\;\dots}_{\dots\,k\,\dots}\right) = \sum_{k=1}^{2} g^{ik}\,\nabla_t A^{\dots\;\;\dots}_{\dots\,k\,\dots}.
\end{aligned} \tag{4.21}$$

Then from the formulas (4.21) we derive the following fact.

Theorem 4.4. *The operation of inner parallel translation of tensors commutate with the operations of index raising and index lowering.*

§ 5. Integration on surfaces. Green's formula.

Let's consider the two-dimensional space $\mathbb{R}^2$. Let's draw it as a coordinate plane u^1, u^2. Let's choose some simply connected domain Ω outlined by a closed piecewise continuously differentiable contour γ on the coordinate plane u^1, u^2. Then we mark the direction (orientation) on the contour γ so that when moving in this direction the domain Ω lies to the left of the contour. On Fig. 5.1 this direction is marked by the arrow. In other words, we choose the orientation on γ induced from the orientation of Ω, i. e. $\gamma = \partial\Omega$.

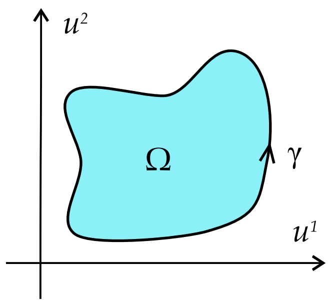

Fig. 5.1

Let's consider a pair of continuously differentiable functions on the coordinate plane: $P(u^1, u^2)$ and $Q(u^1, u^2)$. Then, if all the above conditions are fulfilled, there is the following integral identity:

$$\oint\limits_{\gamma} (P\,du^1 + Q\,du^2) = \iint\limits_{\Omega} \left(\frac{\partial Q}{\partial u^1} - \frac{\partial P}{\partial u^2}\right) du^1 du^2. \tag{5.1}$$

The identity (5.1) is known as Green's formula (see [2]). The equality (5.1) is an equality for a plane. We need its generalization for the case of an arbitrary surface in the space $\mathbb{E}$. In such generalization the coordinate plane u^1, u^2 or some its part plays the role of a chart, while the real geometric domain and its boundary contour should be placed on a surface. Therefore, the integrals in both parts of Green's formula should be transformed so that one can easily write them for any curvilinear coordinates on a surface and their values should not depend on a particular choice of such coordinate system.

Let's begin with the integral in the left hand side of (5.1). Such integrals are called *path integrals of the second kind*. Let's rename P to v_1 and Q to v_2. Then the integral in the left hand side of (5.1) is written as

$$I = \oint\limits_{\gamma} \sum_{i=1}^{2} v_i(u^1, u^2)\,du^i. \tag{5.2}$$

In order to calculate the integral (5.2) practically the contour γ should be parametrized, i. e. it should be represented as a parametric curve (1.1). Then the value of an integral of the second kind is calculated as follows:

$$I = \pm\int\limits_{a}^{b} \left(\sum_{i=1}^{2} v_i\,\dot{u}^i\right) dt. \tag{5.3}$$

This formula reducing the integral of the second kind to the regular integral over the segment $[a,\, b]$ on the real axis can be taken for the definition of the integral

(5.2). The sign is chosen regarding to the direction of the contour on Fig. 5.1. If $a < b$ and if when t changes from a to b the corresponding point on the contour moves along the arrow, we choose plus in (5.3). Otherwise, we choose minus. Changing the variable $\tilde{t} = \varphi(t)$ in the integral (5.3) and choosing the proper sign upon reparametrization of the contour, one can verify that the value of this integral does not depend on the choice of the parametrization on the contour.

Now let's change the curvilinear coordinate system on the surface. The derivatives $\dot{u}^i$ in the integral (5.3) under a change of curvilinear coordinates on the surface are transformed as follows:

$$\dot{u}^i = \frac{du^i}{dt} = \sum_{j=1}^{2} \frac{\partial u^i}{\partial \tilde{u}^j} \frac{d\tilde{u}^j}{dt} = \sum_{j=1}^{2} S^i_j \, \dot{\tilde{u}}^j. \tag{5.4}$$

Substituting (5.4) into the formula (5.3), for the integral I we derive:

$$I = \pm \int\limits_a^b \left(\sum_{j=1}^{2} \left(\sum_{i=1}^{2} S^i_j \, v_i \right) \dot{\tilde{u}}^j \right) dt. \tag{5.5}$$

Now let's write the relationship (5.3) in coordinates $\tilde{u}^1$, $\tilde{u}^2$. For this purpose we rename u^i to $\tilde{u}^i$ and v_i to $\tilde{v}_i$ in the formula (5.3):

$$I = \pm \int\limits_a^b \left(\sum_{i=1}^{2} \tilde{v}_i \, \dot{\tilde{u}}^i \right) dt. \tag{5.6}$$

Comparing the formulas (5.5) and (5.6), we see that these formulas are similar in their structure. For the numeric values of the integrals (5.3) and (5.6) to be always equal (irrespective to the form of the contour γ and its parametrization) the quantities v_i and $\tilde{v}_i$ should be related as follows:

$$\tilde{v}_j = \sum_{i=1}^{2} S^i_j \, v_i, \qquad\qquad v_i = \sum_{i=1}^{2} T^j_i \, \tilde{v}_j.$$

These formulas represent the transformation rule for the components of a covectorial field. Thus, we conclude that any path integral of the second kind on a surface (5.2) is given by some inner covectorial field on this surface.

Now let's proceed with the integral in the right hand side of the Green's formula (5.1). Distracting for a while from the particular integral in this formula, let's consider the following double integral:

$$I = \iint\limits_{\Omega} F \; du^1 du^2. \tag{5.7}$$

A change of curvilinear coordinates can be interpreted as a change of variables in the integral (5.7). Remember that a change of variables in a multiple integral is performed according to the following formula (see [2]):

$$\iint\limits_{\Omega} F \; du^1 du^2 = \iint\limits_{\tilde{\Omega}} F \, |\det J| \; d\tilde{u}^1 d\tilde{u}^2, \tag{5.8}$$

where J is the Jacobi matrix determined by the change of variables:

$$J = \left\| \begin{matrix} \dfrac{\partial u^1}{\partial \tilde{u}^1} & \dfrac{\partial u^1}{\partial \tilde{u}^2} \\ \dfrac{\partial u^2}{\partial \tilde{u}^1} & \dfrac{\partial u^2}{\partial \tilde{u}^2} \end{matrix} \right\|. \tag{5.9}$$

The Jacobi matrix (5.9) coincides with the transition matrix S (see formula (2.7) in Chapter IV). Therefore, the function F being integrated in the formula (5.7) should obey the transformation rule

$$\tilde{F} = |\det S|\, F \tag{5.10}$$

under a change of curvilinear coordinates on the surface. The quantity F has no indices. However, due to (5.10), this quantity is neither a scalar nor a pseudoscalar. In order to change this not very pleasant situation the integral (5.7) over a two-dimensional domain Ω on a surface is usually written as

$$I = \iint\limits_{\Omega} \sqrt{\det \mathbf{g}}\; f\; du^1 du^2, \tag{5.11}$$

where $\det \mathbf{g}$ is the determinant of the first quadratic form. In this case the quantity f in the formula (5.11) is a scalar. This fact follows from the equality $\det \mathbf{g} = (\det T)^2 \det \tilde{\mathbf{g}}$ that represent the transformation rule for the determinant of the metric tensor under a change of coordinate system.

Returning back to the integral in the right hand side of (5.1), we transform it to the form (5.11). For this purpose we use the above notations $P = v_1$, $Q = v_2$, and remember that v_1 and v_2 are the components of the covectorial field. Then

$$\frac{\partial Q}{\partial u^1} - \frac{\partial P}{\partial u^2} = \frac{\partial v_2}{\partial u^1} - \frac{\partial v_1}{\partial u^2}. \tag{5.12}$$

The right hand side of (5.12) can be represented in form of the contraction with the unit skew-symmetric matrix d^{ij} (see formula (3.6) in Chapter IV):

$$\frac{\partial v_2}{\partial u^1} - \frac{\partial v_1}{\partial u^2} = \sum_{i=1}^{2} \sum_{j=1}^{2} d^{ij}\, \frac{\partial v_j}{\partial u^i} = \sum_{i=1}^{2} \frac{\partial}{\partial u^i} \left(\sum_{j=1}^{2} d^{ij}\, v_j \right). \tag{5.13}$$

Note that the quantities d_{ij} with lower indices enter the formula for the area tensor $\boldsymbol{\omega}$ (see (3.7) in Chapter IV). Let's raise the indices of the area tensor by means of the inverse metric tensor:

$$\omega^{ij} = \sum_{p=1}^{2} \sum_{q=1}^{2} g^{ip}\, g^{jq}\, \omega_{pq} = \sum_{p=1}^{2} \sum_{q=1}^{2} \xi_D \sqrt{\det \mathbf{g}}\; g^{ip}\, g^{jq}\, d_{pq}.$$

Applying the formula (3.7) from Chapter IV, we can calculate the components of the area tensor ω^{ij} in the explicit form:

$$\omega^{ij} = \xi_D \sqrt{\det \mathbf{g}^{-1}}\, d^{ij}. \tag{5.14}$$

The formula (5.14) expresses ω^{ij} through d^{ij}. Now we use (5.14) in order to express d^{ij} in the formula (5.13) back through the components of the area tensor:

$$\frac{\partial v_2}{\partial u^1}-\frac{\partial v_1}{\partial u^2}=\sum_{i=1}^{2}\frac{\partial}{\partial u^i}\left(\sum_{j=1}^{2}\xi_D\,\sqrt{\det\mathbf{g}}\;\omega^{ij}\,v_j\right).$$

In order to simplify the further calculations we denote

$$y^i=\sum_{j=1}^{2}\omega^{ij}\,v_j. \tag{5.15}$$

Taking into account (5.15), the formula (5.13) can be written as follows:

$$\begin{aligned}\frac{\partial v_2}{\partial u^1}-\frac{\partial v_1}{\partial u^2}&=\sum_{i=1}^{2}\xi_D\,\frac{\partial\bigl(\sqrt{\det\mathbf{g}}\;y^i\bigr)}{\partial u^i}=\\&=\xi_D\,\sqrt{\det\mathbf{g}}\,\sum_{i=1}^{2}\left(\frac{\partial y^i}{\partial u^i}+\frac{1}{2}\,\frac{\partial\bigl(\ln\det\mathbf{g}\bigr)}{\partial u^i}\,y^i\right).\end{aligned} \tag{5.16}$$

The logarithmic derivative for the determinant of the metric tensor is calculated by means of the lemma 7.1 from Chapter IV. However, we need not repeat these calculations here, since this derivative is already calculated (see (7.12) and the proof of the theorem 7.2 in Chapter IV):

$$\frac{\partial\bigl(\ln\det\mathbf{g}\bigr)}{\partial u^i}=\sum_{p=1}^{2}\sum_{q=1}^{2}g^{pq}\,\frac{\partial g_{pq}}{\partial u^i}=\sum_{q=1}^{2}2\,\Gamma^q_{iq}. \tag{5.17}$$

With the use of (5.17) the formula (5.16) is transformed as follows:

$$\frac{\partial v_2}{\partial u^1}-\frac{\partial v_1}{\partial u^2}=\xi_D\,\sqrt{\det\mathbf{g}}\,\sum_{i=1}^{2}\left(\frac{\partial y^i}{\partial u^i}+\sum_{q=1}^{2}\Gamma^q_{qi}\,y^i\right).$$

In this formula one easily recognizes the contraction of the covariant differential of the vector field $\mathbf{y}$. Indeed, we have

$$\frac{\partial v_2}{\partial u^1}-\frac{\partial v_1}{\partial u^2}=\xi_D\,\sqrt{\det\mathbf{g}}\,\sum_{i=1}^{2}\nabla_i y^i. \tag{5.18}$$

Using the formula (5.18), the notations (5.15), and the autoparallelism condition for the area tensor $\nabla_q\omega^{ij}=0$, we can write the Green's formula as

$$\oint\limits_{\gamma}\sum_{i=1}^{2}v_i\,du^i=\xi_D\iint\limits_{\Omega}\sum_{i=1}^{2}\sum_{j=1}^{2}\omega^{ij}\nabla_i v_j\,\sqrt{\det\mathbf{g}}\,du^1du^2. \tag{5.19}$$

The sign factor ξ_D in (5.19) should be especially commented. The condition that the domain Ω should lie to the left of the contour γ when moving along the arrow is not invariant under an arbitrary change of coordinates u^1, u^2 by $\tilde{u}^1$, $\tilde{u}^2$. Indeed, if we set $\tilde{u}^1 = -u^1$ and $\tilde{u}^2 = u^2$, we would have the mirror image of the domain Ω and the contour γ shown on Fig. 5.1. This means that the direction should be assigned to the geometric contour γ lying on the surface, not to its image in a chart. Then the sign factor ξ_D in (5.19) can be omitted.

The choice of the direction on a geometric contour outlining a domain on a surface is closely related to the choice of the normal vector on that surface. The normal vector $\mathbf{n}$ should be chosen so that when observing from the end of the vector $\mathbf{n}$ and moving in the direction of the arrow along the contour γ the domain Ω should lie to the left of the contour. The choice of the normal vector $\mathbf{n}$ defines the orientation of the surface thus defining the unit pseudoscalar field ξ_D.

§ 6. Gauss-Bonnet theorem.

Let's consider again the process of inner parallel translation of tangent vectors along curves on surfaces. The equation (4.6) shows that from the outer (three-dimensional) point of view this parallel translation differs substantially from the regular parallel translation: the vectors being translated do not remain parallel to the fixed direction in the space — they change. However, their lengths are preserved, and, if we translate several vectors along the same curve, the angles between vectors are preserved (see theorem 4.3).

From the above description, we see that in the process of parallel translation, apart from the motion of the attachment point along the curve, the rotation of the vectors about the normal vector $\mathbf{n}$ occurs. Therefore, we have the natural problem — how to measure the angle of this rotation? We consider this problem just below.

Suppose that we have a surface equipped with the orientation. This means that the orientation field ξ_D and the area tensor $\boldsymbol{\omega}$ are defined (see formula (3.10) in Chapter IV). We already know that ξ_D fixes one of the two possible normal vectors $\mathbf{n}$ at each point of the surface (see formula (4.3) in Chapter IV).

Theorem 6.1. *The inner tensor field Θ of the type $(1,1)$ with the components*

$$\theta^i_j = \sum_{k=1}^{2} \omega_{jk}\, g^{ki} \tag{6.1}$$

is an operator field describing the counterclockwise rotation in the tangent plane to the angle $\pi/2 = 90^\circ$ about the normal vector $\mathbf{n}$.

Proof. Let $\mathbf{a}$ be a tangent vector to the surface and let $\mathbf{n}$ be the unit normal vector at the point where $\mathbf{a}$ is attached. Then, in order to construct the vector $\mathbf{b} = \Theta(\mathbf{a})$ obtained by rotating $\mathbf{a}$ counterclockwise to the angle $\pi/2 = 90^\circ$ about the vector $\mathbf{n}$ one can use the following vector product:

$$\mathbf{b} = \Theta(\mathbf{a}) = [\mathbf{n},\, \mathbf{a}]. \tag{6.2}$$

Let's substitute the expression given by the formula (4.3) from Chapter IV for the vector $\mathbf{n}$ into (6.2). Then let's expand the vector $\mathbf{a}$ in the basis $\mathbf{E}_1$, $\mathbf{E}_2$:

$$\mathbf{a} = a^1 \cdot \mathbf{E}_1 + a^2 \cdot \mathbf{E}_2. \tag{6.3}$$

As a result for the vector $\mathbf{b}$ in the formula (6.2) we derive

$$\mathbf{b} = \sum_{j=1}^{2} \xi_D \cdot \frac{[[\mathbf{E}_1,\, \mathbf{E}_2],\, \mathbf{E}_j]}{|[\mathbf{E}_1,\, \mathbf{E}_2]|} \cdot a^j. \tag{6.4}$$

In order to calculate the denominator in the formula (6.4) we use the well-known formula from the analytical geometry (see [4]):

$$|[\mathbf{E}_1,\, \mathbf{E}_2]|^2 = \det \begin{vmatrix} (\mathbf{E}_1 \,|\, \mathbf{E}_1) & (\mathbf{E}_1 \,|\, \mathbf{E}_2) \\ (\mathbf{E}_2 \,|\, \mathbf{E}_1) & (\mathbf{E}_2 \,|\, \mathbf{E}_2) \end{vmatrix} = \det \mathbf{g}.$$

As for the numerator in the formula (6.4), here we use the not less known formula for the double vectorial product:

$$[[\mathbf{E}_1,\, \mathbf{E}_2],\, \mathbf{E}_j] = \mathbf{E}_2 \cdot (\mathbf{E}_1 \,|\, \mathbf{E}_j) - \mathbf{E}_1 \cdot (\mathbf{E}_j \,|\, \mathbf{E}_2).$$

Taking into account these two formulas, we can write (6.4) as follows:

$$\mathbf{b} = \sum_{j=1}^{2} \xi_D \cdot \frac{g_{1j} \cdot \mathbf{E}_2 - g_{2j} \cdot \mathbf{E}_1}{\sqrt{\det \mathbf{g}}} \cdot a^j. \tag{6.5}$$

Using the components of the area tensor (5.14), no we can rewrite (6.5) in a more compact and substantially more elegant form:

$$\mathbf{b} = \sum_{i=1}^{2} \left(\sum_{j=1}^{2} \sum_{k=1}^{2} \omega^{ki} \, g_{kj} \, a^j \right) \cdot \mathbf{E}_i.$$

From this formula it is easy to extract the formula (6.1) for the components of the linear operator Θ relating $\mathbf{b}$ and $\mathbf{a}$. The theorem is proved. $\square$

The operator field Θ is the contraction of the tensor product of two fields $\boldsymbol{\omega}$ and $\mathbf{g}$. The autoparallelism of the latter ones means that Θ is also an autoparallel field, i. e. $\nabla\Theta = 0$.

We use the autoparallelism of Θ in the following way. Let's choose some parametric curve γ on a surface and perform the parallel translation of some unit vector $\mathbf{a}$ along this curve. As a result we get the vector-valued function $\mathbf{a}(t)$ on the curve satisfying the equation of parallel translation $\nabla_t \mathbf{a} = 0$ (see formula (4.8)). Then we define the vector-function $\mathbf{b}(t)$ on the curve as follows:

$$\mathbf{b}(t) = \Theta(\mathbf{a}(t)). \tag{6.6}$$

From (6.6) we derive $\nabla_t(\mathbf{b}) = \nabla_t\Theta(\mathbf{a}) + \Theta(\nabla_t \mathbf{a}) = 0$. This means that the function (6.6) also satisfies the equation of parallel translation. It follows from the autoparallelism of Θ and from the items (2) and (3) in the theorem 4.2. The vector-functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$ determine two mutually perpendicular unit vectors at each point of the curve. There are the following obvious relationships for them:

$$\Theta(\mathbf{a}) = \mathbf{b}, \qquad\qquad \Theta(\mathbf{b}) = -\mathbf{a}. \tag{6.7}$$

Let's remember for the further use that $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are obtained by parallel translation of the vectors $\mathbf{a}(0)$ and $\mathbf{b}(0)$ along the curve from its initial point.

Now let's consider some inner vector field $\mathbf{x}$ on the surface (it is tangent to the surface in the outer representation). If the field vectors $\mathbf{x}(u^1, u^2)$ are nonzero at each point of the surface, they can be normalized to the unit length: $\mathbf{x} \to \mathbf{x}/|\mathbf{x}|$. Therefore, we shall assume $\mathbf{x}$ to be a field of unit vectors: $|\mathbf{x}| = 1$. At the points of the curve γ this field can be expanded in the basis of the vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{x} = \cos(\varphi) \cdot \mathbf{a} + \sin(\varphi) \cdot \mathbf{b}. \tag{6.8}$$

The function $\varphi(t)$ determines the angle between the vector $\mathbf{a}$ and the field vector $\mathbf{x}$ measured from $\mathbf{a}$ to $\mathbf{x}$ in the counterclockwise direction. The change of φ describes the rotation of the vectors during their parallel translation along the curve.

Let's apply the covariant differentiation ∇_t to the relationship (6.8) and take into account that both vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy the equation of parallel translation:

$$\nabla_t \mathbf{x} = (-\sin(\varphi) \cdot \mathbf{a} + \cos(\varphi) \cdot \mathbf{b}) \cdot \dot{\varphi}. \tag{6.9}$$

Here we used the fact that the covariant derivative ∇_t for the scalar coincides with the regular derivative with respect to t. In particular, we have $\nabla_t \varphi = \dot{\varphi}$. Now we apply the operator Θ to both sides of (6.8) and take into account (6.7):

$$\Theta(\mathbf{x}) = \cos(\varphi) \cdot \mathbf{b} - \sin(\varphi) \cdot \mathbf{a}. \tag{6.10}$$

Now we calculate the scalar product of $\Theta(\mathbf{x})$ from (6.10) and $\nabla_t \mathbf{x}$ from (6.9). Remembering that $\mathbf{a}$ and $\mathbf{b}$ are two mutually perpendicular unit vectors, we get

$$(\Theta(\mathbf{x}) \,|\, \nabla_t \mathbf{x}) = (\cos^2(\varphi) + \sin^2(\varphi))\, \dot{\varphi} = \dot{\varphi}. \tag{6.11}$$

Let's write the equality (6.11) in coordinate form. The vector-function $\mathbf{x}(t)$ on the curve is the restriction of the vector field $\mathbf{x}$, therefore, the covariant derivative $\nabla_t \mathbf{x}$ is the contraction of the covariant differential $\nabla \mathbf{x}$ with the tangent vector of the curve (see formula (4.19)). Hence, we have

$$\dot{\varphi} = \sum_{q=1}^{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \left(x^i\, \omega_{ij}\, \nabla_q x^j\right) \dot{u}^q. \tag{6.12}$$

Here in deriving (6.12) from (6.11) we used the formula (6.1) for the components of the operator field Θ.

Let's discuss the role of the field $\mathbf{x}$ in the construction described just above. The vector field $\mathbf{x}$ is chosen as a reference mark relative to which the rotation angle of the vector $\mathbf{a}$ is measured. This way of measuring the angle is relative. Changing the field $\mathbf{x}$, we would change the value of the angle φ. We have to admit this inevitable fact since tangent planes to the surface at different points are not parallel to each other and we have no preferable direction relative to which we could measure the angles on all of them.

There is a case where we can exclude the above uncertainty of the angle. Let's consider a closed parametric contour γ on the surface. Let $[0, 1]$ be the range over which the parameter t runs on such contour. Then $\mathbf{x}(0)$ and $\mathbf{x}(1)$ do coincide. They represent the same field vector at the point with coordinates $u^1(0), u^2(0)$:

$$\mathbf{x}(0) = \mathbf{x}(1) = \mathbf{x}(u^1(0), u^2(0)).$$

Unlike $\mathbf{x}(t)$, the function $\mathbf{a}(t)$ is not the restriction of a vector field to a curve γ. Therefore, the vectors $\mathbf{a}(0)$ and $\mathbf{a}(1)$ can be different. This is an important feature of the inner parallel translation that differs it from the parallel translation in the Euclidean space $\mathbb{E}$.

In the case of a closed contour γ the difference $\varphi(1) - \varphi(0)$ characterizes the angle to which the vector $\mathbf{a}$ turns $\mathbf{a}$ as a result of parallel translation along the contour. Note that measuring the angle from $\mathbf{x}$ to $\mathbf{a}$ is opposite to measuring it from $\mathbf{a}$ to $\mathbf{x}$ in the formula (6.8). Therefore, taking for positive the angle measured from $\mathbf{x}$ in the counterclockwise direction, we should take for the increment of the angle gained during the parallel translation along γ the following quantity:

$$\triangle\varphi = \varphi(0) - \varphi(1) = -\int\limits_0^1 \dot{\varphi}\,dt.$$

Let's substitute (6.12) for $\dot{\varphi}$ into this formula. As a result we get

$$\triangle\varphi = -\int\limits_0^1 \left(\sum_{q=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\bigl(x^i\,\omega_{ij}\,\nabla_{q}x^j\bigr)\,\dot{u}^q\right)dt. \tag{6.13}$$

Comparing (6.13) with (5.3), we see that (6.13) now can be written in the form of a path integral of the second kind:

$$\triangle\varphi = -\oint\limits_{\gamma}\sum_{q=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\bigl(x^i\,\omega_{ij}\,\nabla_{q}x^j\bigr)\,du^q. \tag{6.14}$$

Assume that the contour γ outlines some connected and simply connected fragment Ω on the surface. Then for this fragment Ω we can apply to (6.14) the Green's formula written in the form of (5.19):

$$\triangle\varphi = -\xi_D\iint\limits_{\Omega}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2}\omega^{ij}\;\nabla_{i}\bigl(x^p\,\omega_{pq}\,\nabla_{j}x^q\bigr)\sqrt{\det\mathbf{g}}\,du^1du^2.$$

If the direction of the contour is in agreement with the orientation of the surface, then the sign factor ξ_D can be omitted:

$$\begin{gathered}\triangle\varphi = -\iint\limits_{\Omega}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2}\bigl(x^p\,\omega^{ij}\omega_{pq}\,\nabla_{i}\nabla_{j}x^q\,+\\ +\,\nabla_{i}x^p\,\omega^{ij}\omega_{pq}\,\nabla_{j}x^q\bigr)\sqrt{\det\mathbf{g}}\,du^1du^2.\end{gathered} \tag{6.15}$$

Let's show that the term $\nabla_i x^p\,\omega^{ij}\omega_{pq}\,\nabla_j x^q$ in (6.15) yields zero contribution to the value of the integral. This feature is specific to the two-dimensional case where we have the following relationship:

$$\omega^{ij}\omega_{pq}=d^{ij}\,d_{pq}=\delta^i_p\,\delta^j_q-\delta^i_q\,\delta^j_p. \tag{6.16}$$

The proof of the formula (6.16) is analogous to the proof of the formula (8.23) in Chapter IV. It is based on the skew-symmetry of d^{ij} and d_{pq}.

Let's complete the inner vector field $\mathbf{x}$ of the surface by the other inner vector field $\mathbf{y}=\Theta(\mathbf{x})$. The vectors $\mathbf{x}$ and $\mathbf{y}$ form a pair of mutually perpendicular unit vectors in the tangents plane. For their components we have

$$\sum_{q=1}^{2} x^q\,x_q=1, \qquad x_i=\sum_{k=1}^{2} g_{ik}\,x^k, \qquad y_i=\sum_{k=1}^{2} g_{ik}\,y^k, \tag{6.17}$$

$$\sum_{q=1}^{2} \nabla_k x^q\,x_q=0, \qquad y_q=\sum_{p=1}^{2} \omega_{pq}\,x^p, \qquad y^i=\sum_{j=1}^{2} \omega^{ji}\,x_j. \tag{6.18}$$

The first relationship (6.17) expresses the fact that $|\mathbf{x}|=1$, other two relationships (6.17) determine the covariant components x_i and y_i of $\mathbf{x}$ and $\mathbf{y}$. The first relationship (6.18) is obtained by differentiating (6.17), the second and the third relationships (6.18) express the vectorial relationship $\mathbf{y}=\Theta(\mathbf{x})$.

Let's multiply (6.16) by $\nabla_k x^q\,x_j\,x^p$ and then sum up over q, p, and j taking into account the relationships (6.17) and (6.18):

$$\nabla_k x^i=\left(\sum_{q=1}^{2} y_q\,\nabla_k x^q\right) y^i=z_k\,y^i. \tag{6.19}$$

Using (6.19), i.e. substituting $\nabla_i x^p=z_i\,y^p$ and $\nabla_j x^q=z_j\,y^q$ into (6.15), we see that the contribution of the second term in this formula is zero. Then, applying (6.16) to (6.15), for the increment $\triangle\varphi$ we derive

$$\triangle\varphi=-\iint\limits_{\Omega}\sum_{i=1}^{2}\sum_{j=1}^{2} x^i\bigl(\nabla_i\nabla_j x^j-\nabla_j\nabla_i x^j\bigr)\,\sqrt{\det\mathbf{g}}\,du^1du^2.$$

Now we apply the relationship (8.5) from Chapter IV to the field $\mathbf{x}$. Moreover, we take into account the formulas (8.24) and (9.9) from Chapter IV:

$$\triangle\varphi=\iint\limits_{\Omega}\sum_{i=1}^{2}\sum_{j=1}^{2}\bigl(K\,g_{ij}\,x^i\,x^j\bigr)\,\sqrt{\det\mathbf{g}}\,du^1du^2.$$

Remember that the vector field $\mathbf{x}$ was chosen to be of the unit length from the very beginning. Therefore, upon summing up over the indices i and j we shall have only the Gaussian curvature under the integration:

$$\triangle\varphi=\iint\limits_{\Omega} K\,\sqrt{\det\mathbf{g}}\,du^1du^2. \tag{6.20}$$

Now let's consider some surface on which a connected and simply connected domain Ω outlined by a piecewise continuously differentiable contour γ is given (see Fig. 6.1). In other words, we have a polygon with curvilinear sides on the surface. The Green's formula (5.1) is applicable to a a piecewise continuously differentiable contour, therefore, the formula (6.20) is valid in this case. The parallel translation of the vector $\mathbf{a}$ along a piecewise continuously differentiable contour is performed step by step. The result of translating the vector $\mathbf{a}$ along a side of the curvilinear polygon γ is used as the initial data for the equations of parallel translation on the succeeding side. Hence, $\varphi(t)$ is a continuous function, though its derivative can be discontinuous at the corners of the polygon.

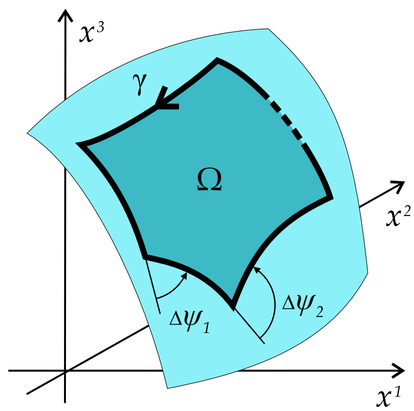

Fig. 6.1

Let's introduce the natural parametrization $t = s$ on the sides of the polygon γ. Then we have the unit tangent vector $\boldsymbol{\tau}$ on them. The vector-function $\boldsymbol{\tau}(t)$ is a continuous function on the sides, except for the corners, where $\boldsymbol{\tau}(t)$ abruptly turns to the angles $\triangle\psi_1$, $\triangle\psi_2$, ... , $\triangle\psi_n$ (see Fig. 6.1). Denote by $\psi(t)$ the angle between the vector $\boldsymbol{\tau}(t)$ and the vector $\mathbf{a}(t)$ being parallel translated along γ. We measure this angle from $\mathbf{a}$ to $\boldsymbol{\tau}$ taking for positive the counterclockwise direction. The finction $\psi(t)$ is a continuously differentiable function on γ except for the corners. At these points it has jump discontinuities with jumps $\triangle\psi_1$, $\triangle\psi_2$, ... , $\triangle\psi_n$.

Let's calculate the derivative of the function $\psi(t)$ out of its discontinuity points. Applying the considerations associated with the expansions (6.8) and (6.9) to the vector $\boldsymbol{\tau}(t)$, for such derivative we find:

$$\dot{\psi} = (\Theta(\boldsymbol{\tau}) \,|\, \nabla_t \boldsymbol{\tau}). \tag{6.21}$$

Then let's calculate the components of the vector $\nabla_t \boldsymbol{\tau}$ in the inner representation of the surface (i. e. in the basis of the frame vectors $\mathbf{E}_1$ and $\mathbf{E}_2$):

$$\nabla_t \tau^k = \ddot{u}^k + \sum_{i=1}^{2} \sum_{j=1}^{2} \Gamma^k_{ji}\, \dot{u}^i\, \dot{u}^j. \tag{6.22}$$

Keeping in mind that $t = s$ is the natural parameter on the sides of the polygon γ, we compare (6.22) with the formula (2.5) for the geodesic curvature and with the formula (2.4). As a result we get the equality

$$\nabla_t \boldsymbol{\tau} = k \cdot \mathbf{n}_{\,\text{curv}} - k_{\,\text{norm}} \cdot \mathbf{n} = k_{\,\text{geod}} \cdot \mathbf{n}_{\,\text{inner}}. \tag{6.23}$$

But $\mathbf{n}_{\,\text{inner}}$ is a unit vector in the tangent plane perpendicular to the vector $\boldsymbol{\tau}$. The same is true for the vector $\Theta(\boldsymbol{\tau})$ in the scalar product (6.21). Hence, the unit vectors $\mathbf{n}_{\,\text{inner}}$ and $\Theta(\boldsymbol{\tau})$ are collinear. Let's denote by $\varepsilon(t)$ the sign factor equal to the scalar product of these vectors:

$$\varepsilon = (\Theta(\boldsymbol{\tau})\,|\,\mathbf{n}_{\,\text{inner}}) = \pm 1. \tag{6.24}$$

Now from the formulas (6.23) and (6.24) we derive:

$$\dot{\psi} = \varepsilon\, k_{\,\text{geod}}. \tag{6.25}$$

Let's find the increment of the function $\psi(t)$ gained as a result of round trip along the whole contour. It is composed by two parts: the integral of (6.25) and the sum jumps at the corners of the polygon γ:

$$\triangle\psi = \oint\limits_{\gamma} \varepsilon\, k_{\,\text{geod}}\, ds + \sum_{i=1}^{n} \triangle\psi_i. \tag{6.26}$$

The angle $\triangle\varphi$ is measured from $\mathbf{x}$ to $\mathbf{a}$ in the counterclockwise direction, while the angle $\triangle\psi$ is measured from $\mathbf{a}$ to $\boldsymbol{\tau}$ in the same direction. Therefore, the sum $\triangle\varphi + \triangle\psi$ is the total increment of the angle between $\mathbf{x}$ and $\boldsymbol{\tau}$. It is important to note that the initial value and the final value of the vector $\boldsymbol{\tau}$ upon round trip along the contour do coincide. The same is true for the vector $\mathbf{x}$. Hence, the sum of increments $\triangle\varphi + \triangle\psi$ is an integer multiple of the angle $2\pi = 360^\circ$:

$$\triangle\varphi + \triangle\psi = 2\pi\, r. \tag{6.27}$$

Practically, the value of the number r in the formula (6.27) is equal to unity. Let's prove this fact by means of the following considerations: we perform the continuous deformation of the surface on Fig. 6.1 flattening it to a plain, then we continuously deform the contour γ to a circle. During such a continuous deformation the left hand side of the equality (6.27) changes continuously, while the right hand side can change only in discrete jumps. Therefore, under the above continuous deformation of the surface and the contour both sides of (6.27) do not change at all. On a circle the total angle of rotation of the unit tangent vector is calculated explicitly, it is equal to 2π. Hence, $r = 1$. We take into account this circumstance when substituting (6.20) and (6.26) into the formula (6.27):

$$\iint\limits_{\Omega} K\,\sqrt{\det\mathbf{g}}\,du^1du^2 + \oint\limits_{\gamma} \varepsilon\, k_{\,\text{geod}}\, ds + \sum_{i=1}^{n} \triangle\psi_i = 2\pi. \tag{6.28}$$

The formula (6.28) is the content of the following theorem which is known as the *Gauss-Bonnet theorem*.

Theorem 6.2. *The sum of the external angles of a curvilinear polygon on a surface is equal to 2π minus two integrals: the area integral of the Gaussian curvature over the interior of the polygon and the integral of the geodesic curvature (taken with the sign factor ε) over its perimeter.*

It is interesting to consider the case where the polygon is formed by geodesic lines on a surface of the constant Gaussian curvature. The second integral in (6.28) then is equal to zero, while the first integral is easily calculated. For the sum of internal angles of a geodesic triangle in this case we derive

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + K\,S,$$

where $K\,S$ is the product of the Gaussian curvature of the surface and the area of the triangle.

A philosophic remark. By measuring the sum of angles of some sufficiently big triangle we can decide whether our world is flat or it is equipped with the curvature. This is not a joke. The idea of a curved space became generally accepted in the modern notions on the structure of the world.

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[1] The references [5] and [6] are added in 2004.