Introduction to Computer Graphics (CS602)
Lecture 10
Mathematics Fundamentals

10.1 Matrices and Simple Matrix Operations

In many fields matrices are used to represent objects and operations on those objects. In computer graphics matrices are heavily used especially their major role is in case of transformations (we will discuss in very next lecture), but not only transformation there are many areas where we use matrices and we will see in what way matrices help us. Anyhow today we are going to discuss matrix and their operation so that we will not face any problem using matrices in coming lectures and in later lectures. Today we will cover following topics:

What a Matrix is?
Dimensions of a Matrix
Elements of a Matrix
Matrix Addition
Zero Matrix
Matrix Negation
Matrix Subtraction
Scalar multiplication of a matrix
The transpose of a matrix

10.1.1 Definition of Matrix

A matrix is a collection of numbers arranged into a fixed number of rows and columns. Usually the numbers are real numbers. In general, matrices can contain complex numbers but we won't see those here. Here is an example of a matrix with three rows and three columns:

$$\begin{array}{rc} & \text{col 1} \\ \text{row 1} \ldots\ldots & \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & 4.6 \\ 4 & -1 & 0 \end{pmatrix} \end{array}$$

The top row is row 1. The leftmost column is column 1. This matrix is a 3x3 matrix because it has three rows and three columns. In describing matrices, the format is:

rows X columns

Each number that makes up a matrix is called an **element** of the matrix. The elements in a matrix have specific locations.

The upper left corner of the matrix is row 1 column 1. In the above matrix the element at row 1 column 1 is the value 1. The element at row 2 columns 3 is the value 4.6.

10.1.2 Matrix Dimensions

The numbers of rows and columns of a matrix are called its **dimensions**. Here is a matrix with three rows and two columns:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2}$$

Sometimes the dimensions are written off to the side of the matrix, as in the above matrix. But this is just a little reminder and not actually part of the matrix. Here is a matrix with different dimensions. It has two rows and three columns. This is a different "data type" than the previous matrix.

$$\begin{pmatrix} 5.12 \\ -4.08 \\ 0.0 \\ 1.0 \end{pmatrix}_{4 \times 1}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 5 & 3 & 4 \end{pmatrix}_{2 \times 3}$$

Question: What do you suppose a **square matrix** is? Here is an example:

$$\begin{pmatrix} 5 & 4 & 3 \\ -4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix}$$

Answer: The number of rows == the number of columns

10.1.3 Square Matrix

In a square matrix the number of rows equals the number of columns. In computer graphics, square matrices are used for transformations.

A **column matrix** consists of a single column. It is a N x 1 matrix. These notes, and most computer graphics texts, use column matrices to represent geometrical vectors. At left is a 4 x 1 column matrix. A **row matrix** consists of a single row.

A column matrix is also called **column vector** and call a row matrix a **row vector**.

Question: What are square matrices used for?
Answer: Square matrices are used (in computer graphics) to represent geometric transformations.

10.1.4 Names for Matrices

Try to remember that matrix starts from rows never from columns so if order of matrix is 3*2 that means there are three rows and two columns. A matrix can be given a name. In printed text, the name for a matrix is usually a capital letter in bold face, like **A** or **M**. Sometimes as a reminder the dimensions are written to the right of the letter, as in $\mathbf{B_{3x3}}$.

The elements of a matrix also have names, usually a lowercase letter the same as the matrix name, with the position of the element written as a subscript. So, for example, the 3x3 matrix **A** might be written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Sometimes you write **A** = $[a_{ij}]$ to say that the elements of matrix **A** are named a_{ij}.

Question: (Thought Question:) If two matrices contain the same numbers as elements, are the two matrices equal to each other?

Answer: No, to be equal, two matrices must have the same dimensions, and must have the same values in the same positions.

10.1.5 Matrix Equality

For two matrices to be equal, they must have

1. The same dimensions.
2. Corresponding elements must be equal.

In other words, say that $\mathbf{A_{n \times m}} = [a_{ij}]$ and that $\mathbf{B_{p \times q}} = [b_{ij}]$.

Then **A** = **B** if and only if n=p, m=q, and $a_{ij}=b_{ij}$ for all i and j in range.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} =?= \begin{pmatrix} 6 & 4 \\ 5 & 2 \\ 1 & 3 \end{pmatrix}$$

Here are two matrices which <u>are not</u> equal even though they have the same elements.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2} =/= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2 \times 3}$$

10.1.6 Matrix Addition

If two matrices have the same number of rows and same number of columns, then the *matrix sum* can be computed:

If **A** is an MxN matrix, and **B** is also an MxN matrix, then their sum is an MxN matrix formed by adding corresponding elements of **A** and **B**

Here is an example of this:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -1 & 2 \\ 1 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 1 & 7 \\ 4 & 0 \end{pmatrix}$$

Of course, in most practical situations the elements of the matrices are real numbers with decimal fractions, not the small integers often used in examples.

Question: What 3x2 matrix could be added to a second 3x2 matrix without changing that second matrix?

Answer: The 3x2 matrix that has all its elements zero.

10.1.7 Zero Matrix

A zero matrix is one; which has all its elements zero. Here is a 3x3 zero matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3 \times 3} = \mathbf{0}$$

The name of a zero matrix is a boldface zero: **0**, although sometimes people forget to make it bold face. Here is an interesting problem:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 10 & 7.5 \\ 2 & 9 & -3.2 \\ -2 & 0 & 5 \end{pmatrix} = ?$$

Question: Form the above sum. No electronic calculators allowed!
Answer: Of course, the sum is the same as the non-zero matrix.

10.1.8 Rules for Matrix Addition

You should be happy with the following rules of matrix addition. In each rule, the matrices are assumed to all have the same dimensions.

A + **B** = **B** + **A**
A + **0** = **0** + **A** = **A**
0 + **0** = **0**

These look the same as some rules for addition of real numbers. (**Warning!!** Not all rules for matrix math look the same as for real number math.)

The first rule says that matrix addition is *commutative*. This is because ordinary addition is being done on the corresponding elements of the two matrices, and ordinary (real) addition is commutative:

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \\ 12 & 14 & 16 \end{pmatrix} = \begin{pmatrix} 1+0 & 3+2 & 5+4 \\ 7+6 & 9+8 & 11+10 \\ 13+12 & 15+14 & 17+16 \end{pmatrix}$$

$$= \begin{pmatrix} 0+1 & 2+3 & 4+5 \\ 6+7 & 8+9 & 10+11 \\ 12+13 & 14+15 & 16+17 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \\ 12 & 14 & 16 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{pmatrix}$$

Question: Do you think that (**A** + **B**) + **C** = **A** + (**B** + **C**)

Answer: Yes — this is another rule that works like real number math.

10.1.9 Practice with Matrix Addition

Here is another matrix addition problem. Mentally form the sum (or use a scrap of paper):

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} = ?$$

Hint: this problem is not as tedious as it might at first seem.

Question: What is the sum?

Answer: Each element of the 3x3 result is 10.

10.1.10 Multiplication of a Matrix by a Scalar

A matrix can be multiplied by a scalar (by a real number) as follows:

To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.

Here is an example of this. (In this example, the variable *a* is a scalar.)

$$a\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1a & 2a & 3a \\ 4a & 5a & 6a \\ 7a & 8a & 9a \end{pmatrix}$$

Question: Show the result if the scalar *a* in the above is the value -1.
Answer: Each element in the result is the negative of the original, as seen below.

10.1.11 Negative of a Matrix

The negation of a matrix is formed by negating each element of the matrix:

-A = -1**A**

So, for example:

$$-1\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{pmatrix}$$

It will not surprise you that **A** + **(-A)** = **0**

Question: Look at the above fact. Can you think of a way to define *matrix subtraction*?

Answer: It seems like subtraction could be defined as adding a negation of a matrix.

10.1.12 Matrix Subtraction

If **A** and **B** have the same number of rows and columns, then **A** - **B** is defined as **A** + **(-B)**. Usually you think of this as:

To form **A** - **B**, from each element of **A** subtract the corresponding element of **B**.

Here is a partly finished example:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 0 \\ ? & -5 & -2 \\ 0 & 2 & ? \end{pmatrix}$$

Notice in particular the elements in the first row of the answer. The way the result was calculated for the elements in row 1 column 2 is sometimes confusion.

Question: Mentally fill in the two question marks.

Answer:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 0 \\ 0 & -5 & -2 \\ 0 & 2 & -6 \end{pmatrix}$$

10.1.13 Transpose

The **transpose** of a matrix is a new matrix whose rows are the columns of the original (which makes its columns the rows of the original). Here is a matrix and its transpose:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix}^T = \begin{pmatrix} 5 & 4 & 7 \\ 4 & 0 & 10 \\ 3 & 4 & 3 \end{pmatrix}$$

The superscript "T" means "transpose". Another way to look at the transpose is that the element at row r column c if the original is placed at row c column r of the transpose. We will usually work with square matrices, and it is usually square matrices that will be transposed. However, non-square matrices can be transposed, as well:

$$\begin{pmatrix} 5 & 4 \\ 4 & 0 \\ 7 & 10 \\ -1 & 8 \end{pmatrix}^T_{4\times 2} = \begin{pmatrix} 5 & 4 & 7 & -1 \\ 4 & 0 & 10 & 8 \end{pmatrix}_{2\times 4}$$

Question: What is the transpose of:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T_{2\times 3} = \ ?$$

Answer:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T_{2\times 3} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3\times 2}$$

10.1.14 A Rule for Transpose

If a transposed matrix is itself transposed, you get the original back:

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T\right)^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

This illustrates the rule $(\mathbf{A}^T)^T = \mathbf{A}$.

Question: What is the transpose of:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T = ?$$

Answer:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

The transpose of a row matrix is a column matrix. And the transpose of a column matrix is a row matrix.

10.1.15 Rule Summary

Here are some rules that cover what has been discussed. You should check that they seem reasonable, rather than memorize them. For each rule the matrices have the same number of rows and columns.

$\mathbf{A} + \mathbf{0} = \mathbf{A}$	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	$\mathbf{0} + \mathbf{0} = \mathbf{0}$
$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	$(ab)\mathbf{A} = a(b\mathbf{A})$	$a(\mathbf{A} + \mathbf{B}) = a\mathbf{B} + a\mathbf{A}$
$a\mathbf{0} = \mathbf{0}$	$(-1)\mathbf{A} = \mathbf{-A}$	$\mathbf{A} - \mathbf{A} = \mathbf{0}$
$(\mathbf{A}^T)^T = \mathbf{A}$	$\mathbf{0}^T = \mathbf{0}$	

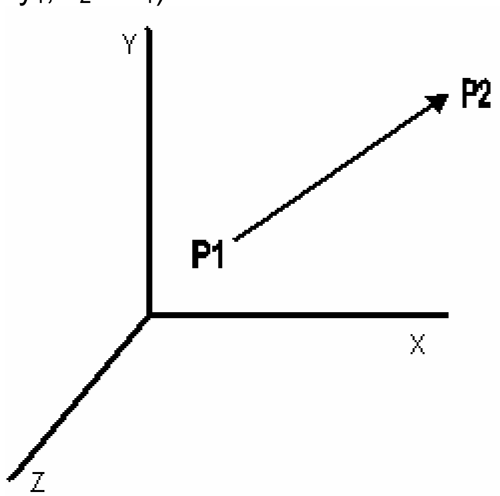
In the above, a and b are scalars (real numbers). **A** and **B** are matrices, and **0** is the zero matrix of appropriate dimension.

Question: If **A** = **B** and **B** = **C**, then does **A** = **C**?

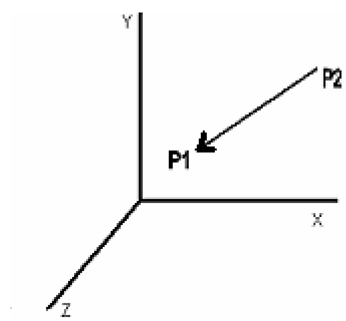
Answer: Yes

10.2 Vectors

Another important mathematical concept used in graphics is the Vector. If $P_1 = (x_1, y_1, z_1)$ is the starting point and $P_2 = (x_2, y_2, z_2)$ is the ending point, then the vector $V = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$



and if $P_2 = (x_2, y_2, z_2)$ is the starting point and $P1 = (x_1, y_1, z_1)$ is the ending point, then the vector $V = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$



This just defines *length and direction*, but *not position*

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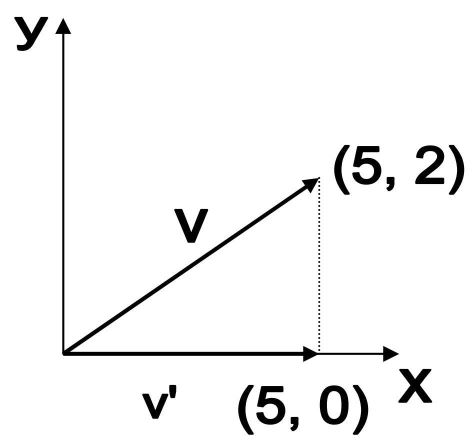
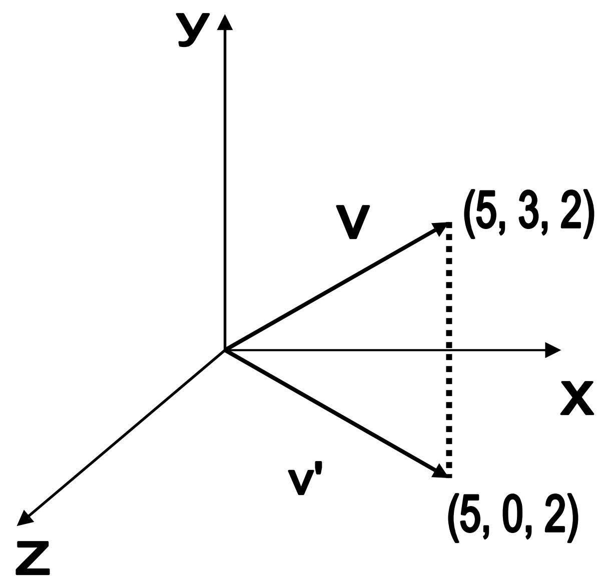
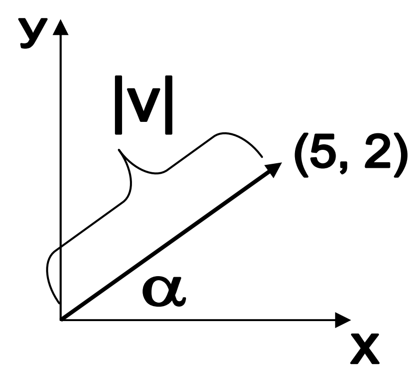
10.2.1 Vector Projections

Projection of v onto the x-axis



Projection of v onto the xz plane





10.2.2 2D Magnitude and Direction

The magnitude (length) of a vector:

$$|V| = \text{sqrt}\ (V_x^2 + V_y^2)$$

The equation is derived from the Pythagorean theorem.
The direction of a vector:

$$\tan \alpha = V_y / V_x$$
$$\alpha = \tan^{-1} (V_y / V_x)$$

Where α is angular displacement from the x-axis.

10.2.3 3D Magnitude and Direction

3D magnitude is a simple extension of 2D

$$|V| = \text{sqrt}(V_x^2 + V_y^2 + V_z^2)$$

3D direction is a bit harder than in 2D. Particularly it needs 2 angles to fully describe direction. Latitude/ longitude is a real-world example.

Direction Cosines are often used:

- α, β, and γ are the positive angles that the vector makes with each positive coordinate axes x, y, and z, respectively

$$\cos \alpha = V_x / |V|$$
$$\cos \beta = V_y / |V|$$
$$\cos \gamma = V_z / |V|$$

10.2.4 Vector Normalization

"Normalizing" a vector means shrinking or stretching it so its magnitude is 1. A simple way is normalize by dividing by its magnitude:

$V = (1, 2, 3)$
$|V| = \text{sqrt}(1^2 + 2^2 + 3^2) = \text{sqrt}(14) + 3.74$

- $V_{norm} = V / |V| = (1, 2, 3) / 3.74 =$
  $(1 / 3.74, 2 / 3.74, 3 / 3.74) = (.27, .53, .80)$

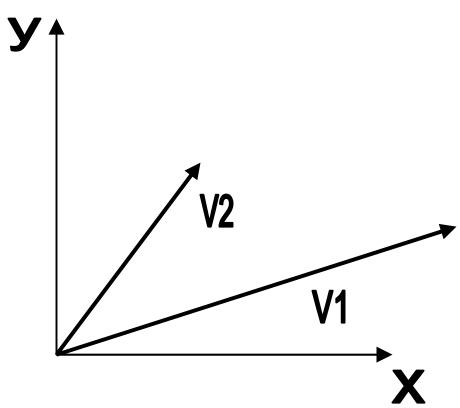
$|V_{norm}| = \text{sqrt}(.27^2 + .53^2 + .80^2) = \text{sqrt}(.9) = .95$

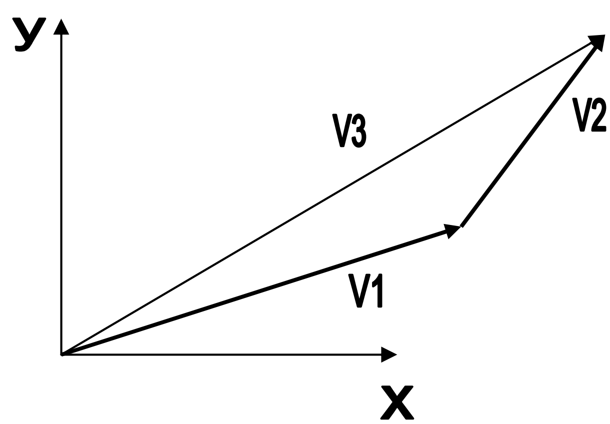
Note that the last calculation doesn't come out to exactly 1. This is because of the error introduced by using only 2 decimal places in the calculations above.

10.2.5 Vector Addition

- Equation:

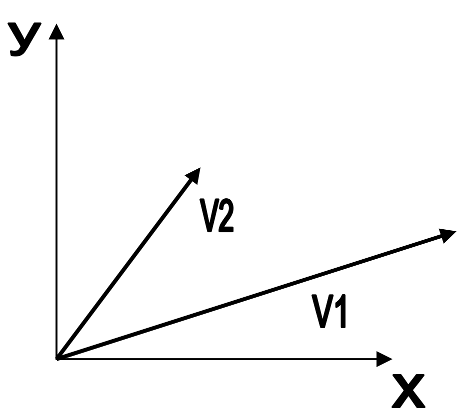
$V_3 = V_1 + V_2 = (V_{1x} + V_{2x}, V_{1y} + V_{2y}, V_{1z} + V_{2z})$

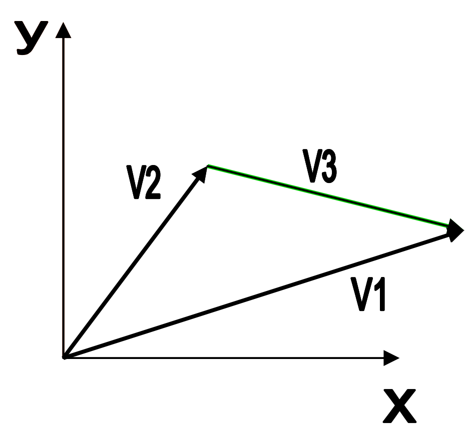




10.2.6 Vector Subtraction

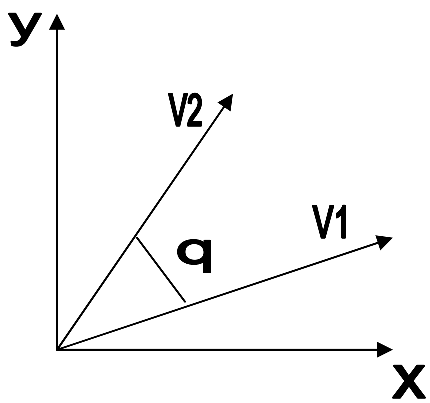
Equation:
$V_3 = V_1 - V_2 = (V_{1x} - V_{2x}, V_{1y} - V_{2y}, V_{1z} - V_{2z})$





10.2.7 Dot Product

The dot product of 2 vectors is a scalar



$V_1 \cdot V_2 = (V_{1x} V_{2x}) + (V_{1y} V_{2y}) + (V_{1z} V_{2z})$

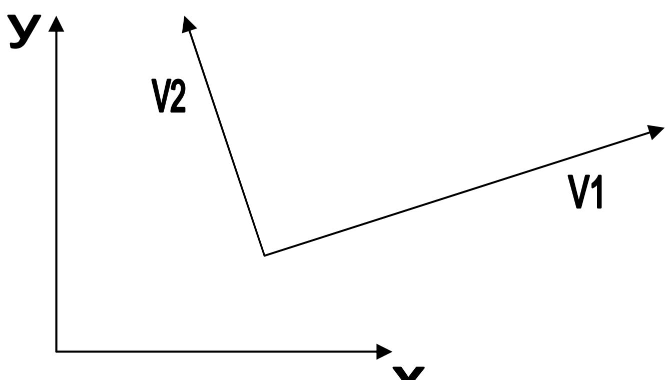
Or, perhaps more importantly for graphics:

$V_1 \cdot V_2 = |V_1|\ |V_2|\ \cos(\theta)$

where θ is the angle between the 2 vectors and θ is in the range $0 \le \theta \le \Pi$

Why is dot product important for graphics?

It is zero if and only if the 2 vectors are perpendicular cos (90) = 0



The Dot Product computation can be simplified when it is known that the vectors are unit vectors

$V_1 \cdot V_2 = \cos(\theta)$

because $|V_1|$ and $|V_2|$ are both 1

Saves 6 squares, 4 additions, and 2 sqrts.

10.2.8 Cross Product

The cross product of 2 vectors is a vector

$V_1 \times V_2 = (V_{1y} V_{2z} - V_{1z} V_{2y},$

$V_{1z} V_{2x} - V_{1x} V_{2z},$

$V_{1x} V_{2y} - V_{1y} V_{2x}$)

Note that if you are big into linear algebra there is also a way to do the cross product calculation using matrices and determinants

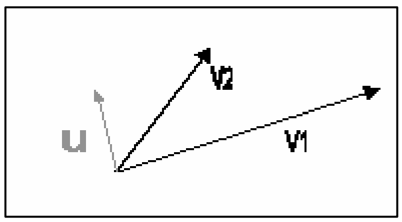
Again, just as with the dot product, there is a more graphical definition:

$V_1 \times V_2 = u\,|V_1|\ |V_2| \sin(\theta)$

where θ is the angle between the 2 vectors and θ is in the range $0 \leq \theta \leq \Pi$ and u is the unit vector that is perpendicular to both vectors

Why u?

$|V_1|\ |V_2| \sin(\theta)$ produces a scalar and the result needs to be a vector.



The direction of u is determined by the right hand rule.

The perpendicular definition leaves an ambiguity in terms of the direction of u
Note that you can’t take the cross product of 2 vectors that are parallel to each other
Sin (0) = sin (180) = 0 à produces the vector (0, 0, 0)

10.2.9 Forming Coordinate Systems

Cross products are great for forming coordinate system frames (3 vectors that are perpendicular to each other) from 2 random vectors.

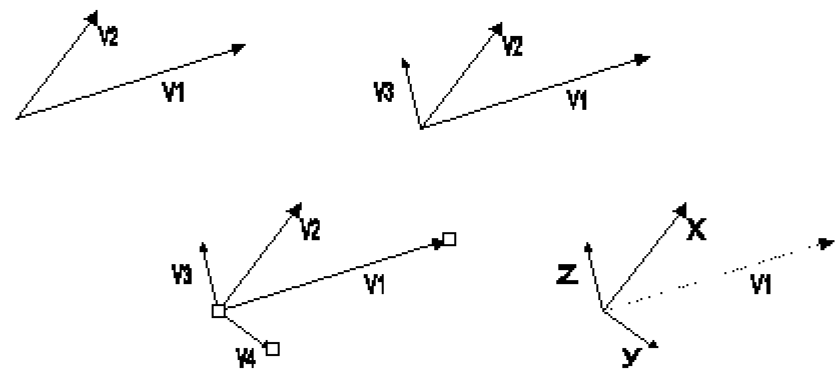
1) Cross V1 and V2 to form V3.

V_3 is now perpendicular to both V_1 and V_2

2) Cross V2 and V3 to form V4

V_4 is now perpendicular to both V_2 and V_3

Then V_2, V_4, and V_3 form your new frame



V_1 and V_2 are in the new xy plane